

A General Modifier-based Framework for Inconsistency-Tolerant Query Answering

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Abstract

We propose a general framework for inconsistency-tolerant query answering within existential rule setting. This framework unifies the main semantics proposed by the state of art and introduces new ones based on cardinality and majority principles. It relies on two key notions: modifiers and inference strategies. An inconsistency-tolerant semantics is seen as a composite modifier plus an inference strategy. We compare the obtained semantics from a productivity point of view.

Introduction

In this paper we place ourselves in the context of Ontology-Based Data Access (Poggi et al. 2008) and we address the problem of query answering when the assertional base (which stores data) is inconsistent with the ontology (which represents generic knowledge about a domain). Existing work in this area studied different inconsistency-tolerant inference relations, called *semantics*, which consist of getting rid of inconsistency by first computing a set of consistent subsets of the assertional base, called *repairs*, that restore consistency w.r.t the ontology, then using them to perform query answering. Most of these proposals, inspired by database approaches *e.g.* (Arenas, Bertossi, and Chomicki 1999) or propositional logic approaches *e.g.* (Benferhat, Dubois, and Prade 1997), were introduced for the lightweight description logic DL-Lite *e.g.* (Lembo et al. 2015). Other description logics *e.g.* (Rosati 2011) or existential rule *e.g.* (Lukasiewicz et al. 2015) have also been considered. In this paper, we use existential rules *e.g.* (Baget et al. 2011) as ontology language that generalizes lightweight description logics.

The main contribution of this paper consists in setting up a general framework that unifies previous proposals and extends the state of the art with new semantics. The idea behind our framework is to distinguish between the way data assertions are virtually distributed (notion of modifiers) and inference strategies. An inconsistency-tolerant semantics is then naturally defined by a modifier and an inference strategy. We also propose a classification of the productivity of

hereby obtained semantics by sound and complete conditions relying on modifier inclusion and inference strategy order. The objective of framework is to establish a methodology for inconsistency handling which, by distinguishing between modifiers and strategies, allows not only to cover existing semantics, but also to easily define new ones, and to study different kinds of their properties.

Preliminaries

We consider first-order logical languages without functional symbols, hence a *term* is a variable or a constant. An *atom* is of the form $p(t_1, \dots, t_k)$ where p is a predicate of arity k , and the t_i are terms. Given an atom or a set of atoms E , $terms(E)$ denotes the set of terms occurring in E . A (factual) *assertion* is an atom without variables.

A *conjunctive query* is an existentially quantified conjunction of atoms. For readability, we restrict our focus to *Boolean* conjunctive queries, which are closed formulas. However the framework and the obtained results can be directly extended to general conjunctive queries. In the following, by *query*, we mean a Boolean conjunctive query. Given a set of assertions \mathcal{A} and a query q , the answer to q over \mathcal{A} is yes iff $\mathcal{A} \models q$, where \models denotes the standard logical consequence.

A knowledge base can be seen as a database enhanced with an ontological component. Since inconsistency-tolerant query answering has been mostly studied in the context of description logics (DLs), and especially DL-Lite, we will use some DL vocabulary, like ABox for the data and TBox for the ontology. However, our framework is not restricted to DLs, hence we define TBoxes and ABoxes in terms of first-order logic. We assume the reader familiar with the basics of DLs and their logical translation.

An *ABox* is a set of factual assertions. As a special case we have DL assertions restricted to unary and binary predicates. A *positive axiom* is of the form $\forall x \forall y (B[x, y] \rightarrow \exists z H[y, z])$ where B and H are conjunctions of atoms (in other words, it is a positive existential rule). As a special case, we have for instance concept and role inclusions in DL-Lite_R, which are respectively of the form $B_1 \sqsubseteq B_2$ and $S_1 \sqsubseteq S_2$, where $B_i := A \mid \exists S$ and $S_i := P \mid P^-$ (with A an atomic concept, P an atomic role and P^- the inverse of an atomic role). A *negative axiom* is of the form $\forall x (B[x] \rightarrow \perp)$ where B is a conjunction of atoms (in other

words, it is a negative constraint). As a special case, we have for instance disjointness axioms in DL-Lite_R, which are inclusions of the form $B_1 \sqsubseteq \neg B_2$ and $S_1 \sqsubseteq \neg S_2$, or equivalently $B_1 \sqcap B_2 \sqsubseteq \perp$ and $S_1 \sqcap S_2 \sqsubseteq \perp$.

A TBox $\mathcal{T} = \mathcal{T}_p \cup \mathcal{T}_n$ is partitioned into a set \mathcal{T}_p of positive axioms and a set \mathcal{T}_n of negative axioms. Finally, a *knowledge base* (KB) is of the form $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ where \mathcal{A} is an ABox and \mathcal{T} is a TBox. \mathcal{K} is said to be *consistent* if $\mathcal{T} \cup \mathcal{A}$ is satisfiable, otherwise it is said to be *inconsistent*. We also say that \mathcal{A} is (in)consistent (with \mathcal{T}), which reflects the assumption that the TBox is reliable. The answer to a query q over a consistent KB \mathcal{K} is yes iff $\langle \mathcal{T}, \mathcal{A} \rangle \models q$. When \mathcal{K} is inconsistent, standard consequence is not appropriate since all queries would be positively answered.

A key notion in inconsistency-tolerant query answering is the one of a repair of the ABox w.r.t. the TBox. A repair is a subset of the ABox consistent with the TBox and inclusion-maximal for this property: $\mathcal{R} \subseteq \mathcal{A}$ is a *repair* of \mathcal{A} w.r.t. \mathcal{T} if i) $\langle \mathcal{T}, \mathcal{R} \rangle$ is consistent, and ii) $\forall \mathcal{R}' \subseteq \mathcal{A}$, if $\mathcal{R} \subsetneq \mathcal{R}'$ (\mathcal{R} is strictly included in \mathcal{R}') then $\langle \mathcal{T}, \mathcal{R}' \rangle$ is inconsistent. We denote by $\mathcal{R}(\mathcal{A})$ the set of \mathcal{A} 's repairs (for easier reading, we often leave \mathcal{T} implicit in our notations). Note that $\mathcal{R}(\mathcal{A}) = \{\mathcal{A}\}$ iff \mathcal{A} is consistent. The most commonly considered semantics for inconsistency-tolerant query answering, inspired from previous work in databases, is the following: q is said to be a *consistent consequence* of \mathcal{K} if it is a standard consequence of each repair of \mathcal{A} . Several variants of this semantics have been proposed, which differ with respect to their behaviour (in particular they can be more or less cautious) and their computational complexity. Before recalling the main semantics studied in the literature, we need to introduce the notion of the positive closure of an ABox. The *positive closure* of \mathcal{A} (w.r.t. \mathcal{T}), denoted by $Cl(\mathcal{A})$, is obtained by adding to \mathcal{A} all assertions (built on the individuals occurring in \mathcal{A}) that can be inferred using the positive axioms of the TBox, namely:

$Cl(\mathcal{A}) = \{A \text{ atom} \mid \langle \mathcal{T}_p, \mathcal{A} \rangle \models A \text{ and } terms(\mathcal{A}) \subseteq terms(A)\}$
Note that the set of atomic consequences of a KB $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ may be infinite whereas the positive closure of \mathcal{A} is always finite since it does not contain new terms. Note also that \mathcal{A} is consistent (with \mathcal{T}) iff $Cl(\mathcal{A})$ is consistent (with \mathcal{T}).

We now recall the most well-known inconsistency-tolerant semantics introduced in (Arenas, Bertossi, and Chomicki 1999; Lembo et al. 2010; Bienvenu 2012). Given a possibly inconsistent KB $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$, a query q is said to be:

- a consistent (or AR) consequence of \mathcal{K} if $\forall \mathcal{R} \in \mathcal{R}(\mathcal{A})$, $\langle \mathcal{T}, \mathcal{R} \rangle \models q$;
- a CAR consequence of \mathcal{K} if $\forall \mathcal{R} \in \mathcal{R}(Cl(\mathcal{A})), \langle \mathcal{T}, \mathcal{R} \rangle \models q$;
- an IAR consequence of \mathcal{K} if $\langle \mathcal{T}, \bigcap_{\mathcal{R} \in \mathcal{R}(\mathcal{A})} \mathcal{R} \rangle \models q$;
- an ICAR consequence of \mathcal{K} if $\langle \mathcal{T}, \bigcap_{\mathcal{R} \in Cl(\mathcal{A})} \mathcal{R} \rangle \models q$;
- an ICR consequence of \mathcal{K} if $\langle \mathcal{T}, \bigcap_{\mathcal{R} \in \mathcal{R}(\mathcal{A})} Cl(\mathcal{R}) \rangle \models q$.

A Unified Framework for Inconsistency-Tolerant Query Answering

In this section, we define a unified framework for inconsistency-tolerant query answering based on two main concepts: modifiers and inference strategies.

Let us first introduce the notion of MBox KBs. While a standard KB has a single ABox, it is convenient for subsequent definitions to define KBs with multiple ABoxes ("MBoxes"). Formally, an *MBox KB* is of the form $\mathcal{K}_{\mathcal{M}} = \langle \mathcal{T}, \mathcal{M} \rangle$ where \mathcal{T} is a TBox and $\mathcal{M} = \{\mathcal{A}_1, \dots, \mathcal{A}_n\}$ is a set of ABoxes called an MBox. We say that $\mathcal{K}_{\mathcal{M}}$ is *consistent*, or \mathcal{M} is consistent (with \mathcal{T}) if each \mathcal{A}_i in \mathcal{M} is consistent (with \mathcal{T}).

In the following, we start with an MBox KB which is a possibly inconsistent standard KB (namely with a single ABox in \mathcal{M}) and produce a consistent MBox KB, in which each element reflects a virtual reparation of the initial ABox. We see an inconsistency-tolerant query answering method as made out of a *modifier*, which produces a consistent MBox from the original ABox (and the Tbox), and an *inference strategy*, which evaluates queries against the obtained MBox KB.

Elementary and Composite Modifiers

We first introduce three classes of elementary modifiers, namely expansion, splitting and selection. For each class, we consider a "natural" instantiation, namely *positive closure*, splitting into *repairs* and selecting the largest elements (i.e., maximal w.r.t. *cardinality*). Elementary modifiers can be combined to define *composite* modifiers. Given the three natural instantiations of these modifiers, we show that their combination yields exactly eight different composite modifiers.

Expansion modifiers. The expansion of an MBox consists in explicitly adding some inferred knowledge to its ABoxes. A natural expansion modifier consists in computing the *positive closure* of an MBox, which is defined as follows:

$$\circ_{cl}(\mathcal{M}) = \{Cl(\mathcal{A}_i) \mid \mathcal{A}_i \in \mathcal{M}\}.$$

Splitting modifiers. A splitting modifier replaces each \mathcal{A}_i of an MBox by one or several of its consistent subsets. A natural splitting modifier consists of splitting each ABox into the set of its repairs, which is defined as follows:

$$\circ_{rep}(\mathcal{M}) = \bigcup_{\mathcal{A}_i \in \mathcal{M}} \{\mathcal{R}(\mathcal{A}_i)\}.$$

This modifier always produces a consistent MBox.

Selection modifiers. A selection modifier selects some subsets of an MBox. As a natural selection modifier, we consider the *cardinality-based selection* modifier, which selects the largest elements of an MBox:

$$\circ_{card}(\mathcal{M}) = \{\mathcal{A}_i \in \mathcal{M} \mid \nexists \mathcal{A}_j \in \mathcal{M} \text{ s.t. } |\mathcal{A}_j| > |\mathcal{A}_i|\}.$$

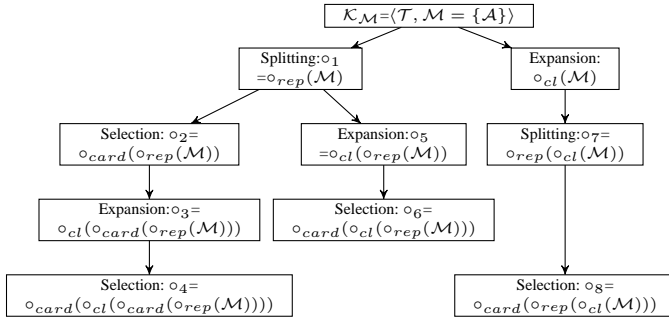


Figure 1: The eight possible combinations of modifiers starting from a single MBox $\mathcal{K}_M = \langle \mathcal{T}, \mathcal{M} = \{A\} \rangle$

We call a *composite modifier* any combination of these three elementary modifiers. We now study the question of how many different composite modifiers yielding consistent MBoxes exist and how they compare to each other. We begin with some properties that considerably reduce the number of combinations to be considered. First, the three modifiers are idempotent. Second, the modifiers \circ_{cl} and \circ_{rep} need to be applied only once.

Lemma 1. *For any MBox \mathcal{M} , the following holds:*

1. $\circ_{cl}(\circ_{cl}(\mathcal{M})) = \circ_{cl}(\mathcal{M})$, $\circ_{rep}(\circ_{rep}(\mathcal{M})) = \circ_{rep}(\mathcal{M})$ and $\circ_{card}(\circ_{card}(\mathcal{M})) = \circ_{card}(\mathcal{M})$.
2. Let \circ_d be any composite modifier. Then:
 - (a) $\circ_{cl}(\circ_d(\circ_{cl}(\mathcal{M}))) = \circ_d(\circ_{cl}(\mathcal{M}))$, and
 - (b) $\circ_{rep}(\circ_d(\circ_{rep}(\mathcal{M}))) = \circ_d(\circ_{rep}(\mathcal{M}))$.

Figure 1 presents the eight different composite modifiers (thanks to Lemma 1) that can be applied to an MBox initially composed of a single (possibly inconsistent) ABox. At the beginning, one can perform either an expansion or a splitting operation (the selection has no effect). Expansion can only be followed by a splitting or a selection operation. From the MBox $\circ_{rep}(\circ_{cl}(\mathcal{M}))$ only a selection can be performed, thanks to Lemma 1. Similarly, if one starts with a splitting operation followed by a selection operation, then only an expansion can be done (thanks to Lemma 1 again). From $\circ_{cl}(\circ_{card}(\circ_{rep}(\mathcal{M})))$ only a selection can be performed (Lemma 1 again).

To ease reading, we also denote the modifiers by short names reflecting the order in which the elementary modifiers are applied, and using the following letters: R for \circ_{rep} , C for \circ_{cl} and M for \circ_{card} as shown in Table 1. For instance, MCR denotes the modifier that first splits the initial ABox into its set of repairs, then closes these repairs and finally selects the maximal-cardinality elements.

Theorem 1. *Let $\mathcal{K}_M = \langle \mathcal{T}, \mathcal{M} = \{A\} \rangle$ be a possibly inconsistent KB. Then for any composite modifier \circ_c that can be obtained by a finite combination of the elementary modifiers \circ_{rep} , \circ_{card} , \circ_{cl} , there exists a composite modifier \circ_i in $\{\circ_1 \dots \circ_8\}$ (see Table 1) such that $\circ_c(\mathcal{M}) = \circ_i(\mathcal{M})$.*

Example 1. Let $\mathcal{K}_M = \langle \mathcal{T}, \mathcal{M} \rangle$ be an MBox DL-Lite KB where $\mathcal{T} = \{A \sqsubseteq \neg B, A \sqsubseteq \neg C, B \sqsubseteq \neg C, A \sqsubseteq D, B \sqsubseteq D, C \sqsubseteq D, B \sqsubseteq E, C \sqsubseteq E\}$ and $\mathcal{M} = \{\{A(a), B(a), C(a), A(b)\}\}$. We have $\circ_1(\mathcal{M}) = \{\{A(a), A(b)\}, \{B(a), A(b)\}, \{C(a), A(b)\}\}$,

Modifier	Combination	MBox
R	$\circ_1 = \circ_{rep}(\cdot)$	$\mathcal{M}_1 = \circ_1(\mathcal{M})$
MR	$\circ_2 = \circ_{card}(\circ_{rep}(\cdot))$	$\mathcal{M}_2 = \circ_2(\mathcal{M})$
CMR	$\circ_3 = \circ_{cl}(\circ_{card}(\circ_{rep}(\cdot)))$	$\mathcal{M}_3 = \circ_3(\mathcal{M})$
MCMR	$\circ_4 = \circ_{card}(\circ_{cl}(\circ_{card}(\circ_{rep}(\cdot))))$	$\mathcal{M}_4 = \circ_4(\mathcal{M})$
CR	$\circ_5 = \circ_{cl}(\circ_{rep}(\cdot))$	$\mathcal{M}_5 = \circ_5(\mathcal{M})$
MCR	$\circ_6 = \circ_{card}(\circ_{cl}(\circ_{rep}(\cdot)))$	$\mathcal{M}_6 = \circ_6(\mathcal{M})$
RC	$\circ_7 = \circ_{rep}(\circ_{cl}(\cdot))$	$\mathcal{M}_7 = \circ_7(\mathcal{M})$
MRC	$\circ_8 = \circ_{card}(\circ_{rep}(\circ_{cl}(\cdot)))$	$\mathcal{M}_8 = \circ_8(\mathcal{M})$

Table 1: The eight possible composite modifiers for an MBox $\mathcal{K}_M = \langle \mathcal{T}, \mathcal{M} = \{A\} \rangle$

$\circ_5(\mathcal{M}) = \{\{A(a), D(a), A(b), D(b)\}, \{B(a), D(a), E(a), A(b), D(b)\}, \{C(a), D(a), E(a), A(b), D(b)\}\}$, and $\circ_6(\mathcal{M}) = \{\{B(a), D(a), E(a), A(b), D(b)\}, \{C(a), D(a), E(a), A(b), D(b)\}\}$.

The composite modifiers can be classified according to "inclusion" as depicted in Figure 2. We consider the relation, denoted by \subseteq_R , defined as follows: given two modifiers X and Y , $X \subseteq_R Y$ if, for any MBox \mathcal{M} , for each $A \in X(\mathcal{M})$ there is $B \in Y(\mathcal{M})$ such that $A \subseteq B$. We also consider two specializations of \subseteq_R : the true inclusion \subseteq (i.e., $X(\mathcal{M}) \subseteq Y(\mathcal{M})$) and the "closure" inclusion, denoted by \subseteq_{cl} : $X \subseteq_{cl} Y$ if $Y(\mathcal{M})$ is the positive closure of $X(\mathcal{M})$ (then each $A \in X(\mathcal{M})$ is included in its closure in $Y(\mathcal{M})$). In Figure 2, there is an edge from a modifier X to a modifier Y iff $X \subseteq_R Y$. We label each edge by the most specific inclusion relation that holds from X to Y . Transitivity edges are not represented.

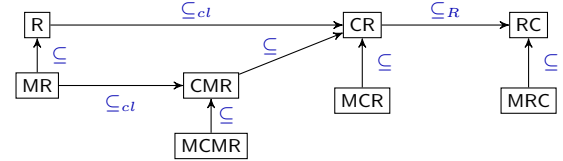


Figure 2: Inclusion relations between composite modifiers.

With any X and Y such that $X \subseteq_R Y$, one can naturally associate, for any MBox \mathcal{M} , a mapping from MBox $X(\mathcal{M})$ to MBox $Y(\mathcal{M})$, which assigns each $A \in X(\mathcal{M})$ to a $B \in Y(\mathcal{M})$ such that $A \subseteq B$. We point out the following useful facts:

Fact 1 The MBox mapping associated with \subseteq_R is injective in all our cases.

Fact 2 The MBox mapping associated with \subseteq_{cl} is surjective (hence bijective). The same holds for the mapping from CR to RC.

Inference Strategies for Querying an MBox

An inference-based strategy takes as input a consistent MBox KB $\mathcal{K}_M = \langle \mathcal{T}, \mathcal{M} \rangle$ and a query q and determines if q is entailed from \mathcal{K}_M . We consider four main inference strategies: universal (also known as skeptical), safe, majority-based and existential (also called brave).¹

¹Of course, one can consider other inference strategies such as the argued inference, parametrized inferences, etc. This is left for future work.

The *universal* inference strategy states that a conclusion is valid iff it is entailed from \mathcal{T} and every ABox in \mathcal{M} . It is a standard way to derive conclusions from conflicting sources, used for instance in default reasoning (Reiter 1980), where one only accepts conclusions derived from each extension of a default theory. The *safe* inference strategy considers as valid conclusions those entailed from \mathcal{T} and the intersection of all ABoxes. The safe inference is a very sound and conservative inference relation since it only considers assertions shared by different ABoxes. The *existential* inference strategy (called also brave inference relation) considers as valid all conclusions entailed from \mathcal{T} and at least one ABox. The existential inference is a very adventurous inference relation and may derive conclusions that are together inconsistent with \mathcal{T} . It is often considered as undesirable when the KB represents available knowledge base on some problem. It only makes sense in some decision problems when one is only looking for a possible solution of a set of constraints or preferences. Finally, the *majority-based* inference relation considers as valid all conclusions entailed from \mathcal{T} and the majority of ABoxes. The majority-based inference can be seen as a good compromise between universal / safe inference and existential inference.

We formally define these inference strategies as follows:

- Query q is a *universal* consequence of $\mathcal{K}_{\mathcal{M}}$, denoted by $\mathcal{K}_{\mathcal{M}} \models_{\forall} q$ iff $\forall A_i \in \mathcal{M}, \langle \mathcal{T}, A_i \rangle \models q$.
- Query q is a *safe* consequence of $\mathcal{K}_{\mathcal{M}}$, denoted by $\mathcal{K}_{\mathcal{M}} \models_{\cap} q$, iff $\langle \mathcal{T}, \bigcap_{A_i \in \mathcal{M}} A_i \rangle \models q$.
- Query q is a *majority-based* consequence of $\mathcal{K}_{\mathcal{M}}$, denoted by $\mathcal{K}_{\mathcal{M}} \models_{maj} q$, iff $\frac{|\{A_i : A_i \in \mathcal{M}, \langle \mathcal{T}, A_i \rangle \models q\}|}{|\mathcal{M}|} > 1/2$.
- Query q is an *existential* consequence of $\mathcal{K}_{\mathcal{M}}$, denoted by $\mathcal{K}_{\mathcal{M}} \models_{\exists} q$ iff $\exists A_i \in \mathcal{M}, \langle \mathcal{T}, A_i \rangle \models q$.

Given two inference strategies s_i and s_j , we say that s_i is *more cautious* than s_j , denoted $s_i \leq s_j$, when for any consistent MBox $\mathcal{K}_{\mathcal{M}}$ and any query q , if $\mathcal{K}_{\mathcal{M}} \models_{s_i} q$ then $\mathcal{K}_{\mathcal{M}} \models_{s_j} q$. The considered inference strategies are totally ordered by \leq as follows:

$$\cap \leq \forall \leq maj \leq \exists \quad (1)$$

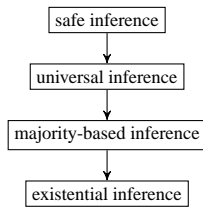


Figure 3: Comparison between inference strategies, where $X \longrightarrow Y$ means that $X \leq Y$

Example 2. Let us consider the MBox $\mathcal{M}_1 = \circ_1(\mathcal{M})$ given in Example 1. We have $\bigcap_{A_i \in \mathcal{M}} A_i = \{A(b)\}$, hence $\mathcal{K}_{\mathcal{M}_1} \models_{\cap} D(b)$. By universal inference, we also have $\mathcal{K}_{\mathcal{M}_1} \models_{\forall} D(a)$. The majority-based inference adds $E(a)$ as a valid conclusion. Indeed, $\langle \mathcal{T}, \{B(a), A(b)\} \rangle \models E(a)$ and $\langle \mathcal{T}, \{C(a), A(b)\} \rangle \models E(a)$ and $|\mathcal{M}_1| = 3$, hence

$\mathcal{K}_{\mathcal{M}_1} \models_{maj} E(a)$. Finally, the existential inference adds $A(a)$ as a valid conclusion.

Inconsistency-Tolerant Semantics = Composite Modifier + Inference Strategy

We can now define an inconsistency-tolerant query answering semantics by a composite modifier and an inference strategy.

Definition 1. Let $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ be a standard KB, \circ_i be a composite modifier and s_j be an inference strategy. A query q is said to be an $\langle \circ_i, s_j \rangle$ -consequence of \mathcal{K} , which is denoted by $\mathcal{K} \models_{\langle \circ_i, s_j \rangle} q$, if it is entailed from the MBox $\mathcal{K} \langle \mathcal{T}, \circ_i(\{\mathcal{A}\}) \rangle$ with the inference strategy s_j .

This definition covers the main semantics recalled in Section : AR, IAR, CAR, ICAR and ICR semantics respectively correspond to $\langle \circ_1, \forall \rangle$, $\langle \circ_1, \cap \rangle$, $\langle \circ_7, \forall \rangle$, $\langle \circ_7, \cap \rangle$ and $\langle \circ_5, \cap \rangle$.

Comparison of Inconsistency-Tolerant Semantics w.r.t. Productivity

We now compare the obtained semantics with respect to productivity, which we formalize as follows.

Definition 2. Given two semantics $\langle \circ_i, s_k \rangle$ and $\langle \circ_j, s_l \rangle$, we say that $\langle \circ_j, s_l \rangle$ is more productive than $\langle \circ_i, s_k \rangle$, and note $\langle \circ_i, s_k \rangle \sqsubseteq \langle \circ_j, s_l \rangle$ if, for any KB $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ and any query q , if $\mathcal{K} \models_{\langle \circ_i, s_k \rangle} q$ then $\mathcal{K} \models_{\langle \circ_j, s_l \rangle} q$.

We first pairwise compare semantics defined with the same inference strategy. For each inference strategy, we give necessary and sufficient conditions for the comparability of the associated semantics w.r.t. productivity. These conditions rely on the inclusion relations between modifiers (see Figure 2).

Proposition 1. [Productivity of \cap -semantics] See Figure 4. It holds that $\langle \circ_i, \cap \rangle \sqsubseteq \langle \circ_j, \cap \rangle$ iff $\circ_j \subseteq \circ_i$ or $\circ_i \subseteq_R \circ_j$ in a bijective way (see Fact 2).

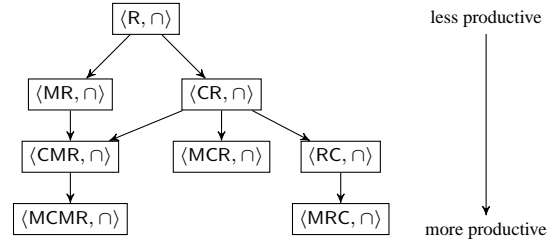


Figure 4: Relationships between \cap -based semantics

Proposition 2. [Productivity of \forall -semantics] See Figure 5. It holds that $\langle \circ_i, \forall \rangle \sqsubseteq \langle \circ_j, \forall \rangle$ iff $\circ_j \subseteq \circ_i$, or $\circ_i \subseteq_R \circ_j$ in a bijective way (see Fact 2) or $\circ_j \subseteq_{cl} \circ_i$.

Proposition 3. [Productivity of *maj*-semantics] See Figure 6. It holds that $\langle \circ_i, maj \rangle \sqsubseteq \langle \circ_j, maj \rangle$ iff $\circ_i \subseteq_R \circ_j$ in a bijective way (see Fact 2) or $\circ_j \subseteq_{cl} \circ_i$.

Proposition 4. [Productivity of \exists -semantics] See Figure 7. It holds that $\langle \circ_i, \exists \rangle \sqsubseteq \langle \circ_j, \exists \rangle$ iff $\circ_i \subseteq_R \circ_j$ (in particular $\circ_i \subseteq \circ_j$ or $\circ_i \subseteq_{cl} \circ_j$) or $\circ_j \subseteq_{cl} \circ_i$.

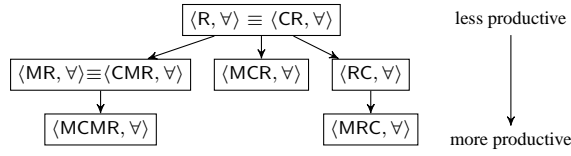


Figure 5: Relationships between \forall -based semantics

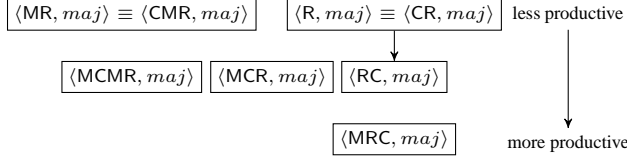


Figure 6: Relationships between *maj*-based semantics

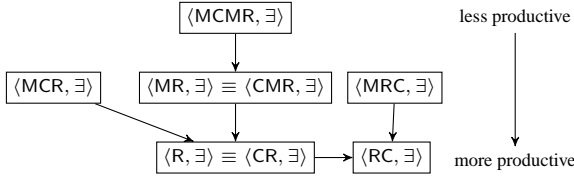


Figure 7: Relationships between \exists -based semantics

We now extend previous results to any pair of semantics, possibly based on different inference strategies.

Theorem 2. [Productivity of semantics] *The inclusion relation \sqsubseteq is the smallest relation that contains the inclusions $\langle \circ_i, s_k \rangle \sqsubseteq \langle \circ_j, s_k \rangle$ defined by Propositions 1-4 and satisfying the two following conditions:*

1. *for all s_j, s_p and \circ_i , if $s_j \leq s_p$ then $\langle \circ_i, s_j \rangle \sqsubseteq \langle \circ_i, s_p \rangle$.*
2. *it is transitive.*

Theorem 2 is an important result. It states that the productivity relation can only be obtained from Figures 4-7 (resp. Propositions 1-4) and some composition of the relations. No more inclusion relations hold. In particular when $s_i > s_j$, it holds that $\forall k, \forall l, \langle \circ_k, s_i \rangle \not\sqsubseteq \langle \circ_l, s_j \rangle$, which means that there exist a query q and a KB \mathcal{K} such that q is an $\langle \circ_k, s_i \rangle$ -consequence of \mathcal{K} but not an $\langle \circ_l, s_j \rangle$ -consequence of \mathcal{K} . Note that this holds already for DL-Lite_R KBs.

Proof:[Sketch] Condition 1 holds by definition of \leq . Transitivity holds by definition of \sqsubseteq . To show that there are no other inclusions, we prove two lemmas: for all $\langle \circ_i, s_j \rangle$ and $\langle \circ_k, s_p \rangle$, (1) if $s_p < s_j$ then $\langle \circ_i, s_j \rangle \not\sqsubseteq \langle \circ_k, s_p \rangle$; and (2) if $\langle \circ_i, s_j \rangle \sqsubseteq \langle \circ_k, s_p \rangle$ and $s_j < s_p$, then $\langle \circ_i, s_p \rangle \sqsubseteq \langle \circ_k, s_p \rangle$. \square

Lastly, it is important to note that when the initial KB is consistent, all semantics collapse with standard entailment, namely:

Proposition 5. *Let \mathcal{K} be a consistent standard KB. Then: $\forall s \in \{\cap, \forall, maj, \exists\}, \forall i : 1, \dots, 8, \mathcal{K} \models_{\langle \circ_i, s \rangle} q$ iff $\mathcal{K} \models q$.*

Conclusion

This paper provides a general and unifying framework for inconsistency-tolerant query answering. On the one hand, our logical setting based on existential rules includes previously considered languages. On the other hand, viewing

an inconsistency-tolerant semantics as a pair composed of a modifier and an inference strategy allows us to include the main known semantics and to consider new ones. We believe that the choice of semantics depends on the applicative context, namely the features of the semantics, i.e rationality properties, complexity (which we have studied, but not presented in this paper) and productivity with respect to the applicative context. In particular, cardinality-based selection allows us to counter troublesome assertions that conflict with many others. In some contexts, requiring to find an answer in all selected repairs can be too restrictive, hence the interest of majority-based semantics, which are more productive than universal semantics, without being as productive as the adventurous existential semantics. As for future work, we plan consider other inference strategies such as the argued inference, parametrized inferences, etc. We also want to adapt the framework to belief change problems, like merging or revision.

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Appendix

In this appendix, we provide details on the proofs.

Section 3: A Unified Framework for Inconsistency-Tolerant Query Answering

Theorem 1. Let $\mathcal{K}_{\mathcal{M}} = \langle \mathcal{T}, \mathcal{M} = \{\mathcal{A}\} \rangle$ be a possibly inconsistent KB. Then for any composite modifier \circ_c that can be obtained by a finite combination of the elementary modifiers \circ_{rep} , \circ_{card} , \circ_{cl} , there exists a composite modifier \circ_i in $\{\circ_1 \dots \circ_8\}$ (see Table 1) such that $\circ_c(\mathcal{M}) = \circ_i(\mathcal{M})$.

Lemma 1 For any MBox \mathcal{M} , the following holds:

1. $\circ_{cl}(\circ_{cl}(\mathcal{M})) = \circ_{cl}(\mathcal{M})$, $\circ_{rep}(\circ_{rep}(\mathcal{M})) = \circ_{rep}(\mathcal{M})$ and $\circ_{card}(\circ_{card}(\mathcal{M})) = \circ_{card}(\mathcal{M})$.
2. Let \circ_d be any composite modifier. Then:
 - (a) $\circ_{cl}(\circ_d(\circ_{cl}(\mathcal{M}))) = \circ_d(\circ_{cl}(\mathcal{M}))$, and
 - (b) $\circ_{rep}(\circ_d(\circ_{rep}(\mathcal{M}))) = \circ_d(\circ_{rep}(\mathcal{M}))$.

Proof. The proof of the idempotence of \circ_{rep} follows from the facts that: i) $\forall \mathcal{A}_i \in \circ_{rep}(\mathcal{M})$, $\langle \mathcal{T}, \mathcal{A}_i \rangle$ is consistent and ii) if $\langle \mathcal{T}, \mathcal{A}_i \rangle$ is consistent, then $\circ_{rep}(\mathcal{A}_i) = \{\mathcal{A}_i\}$. The proof of the idempotence of \circ_{card} follows from the facts that: i) $\forall \mathcal{A}_i \in \circ_{card}(\mathcal{M})$, $\forall \mathcal{A}_j \in \circ_{card}(\mathcal{M})$, we have $|\mathcal{A}_i| = |\mathcal{A}_j|$ ii) if $\forall \mathcal{A}_i \in \circ_{card}(\mathcal{M})$, $\forall \mathcal{A}_j \in \circ_{card}(\mathcal{M})$, $|\mathcal{A}_i| = |\mathcal{A}_j|$ then $\circ_{card}(\mathcal{M}) = \mathcal{M}$. For the idempotence of \circ_{cl} , it is enough to show that for a given $\mathcal{A} \in \mathcal{M}$, $\circ_{cl}(\circ_{cl}(\mathcal{A})) = \circ_{cl}(\mathcal{A})$. From the definition of \circ_{cl} , clearly we have $\circ_{cl}(\mathcal{A}) \subseteq \circ_{cl}(\circ_{cl}(\mathcal{A}))$. Now assume that $f \in \circ_{cl}(\circ_{cl}(\mathcal{A}))$ but $f \notin \circ_{cl}(\mathcal{A})$. Let $B_f \subseteq \circ_{cl}(\mathcal{A})$ be the subset that allows to derive f , namely $\langle \mathcal{T}_p, B_f \rangle \models f$. Now for each element x of B_f , we have $\langle \mathcal{T}_p, \mathcal{A} \rangle \models x$. Then clearly, $\langle \mathcal{T}_p, \mathcal{A} \rangle \models f$.

Regarding item (2.a), if \circ_d is an elementary modifier then it can be either \circ_{cl} , \circ_{card} , or \circ_{rep} . If $\circ_d = \circ_{cl}$ then the result holds since \circ_{cl} is idempotent. If $\circ_d = \circ_{card}$ then the selected elements from $\circ_{card}(\circ_{cl}(\mathcal{M}))$ are closed sets of assertions since \circ_{card} only discards some elements of $\circ_{cl}(\mathcal{M})$ but does not change the content of remaining elements. Lastly, let us consider the case where $\circ_d = \circ_{rep}$. Again $\forall \mathcal{A}' \in \circ_{rep}(\circ_{cl}(\mathcal{M}))$, $\mathcal{A}' = \circ_{cl}(\mathcal{A}')$. Let us recall that \mathcal{A}' is a maximally consistent subset of $\mathcal{A} \in \circ_{cl}(\mathcal{M})$, with $\mathcal{A} = \circ_{cl}(\mathcal{A})$. If $\mathcal{A}' \neq \circ_{cl}(\mathcal{A})$ this means that $\exists f \in \circ_{cl}(\mathcal{A}')$ (hence $f \in \mathcal{A}$) such that $f \notin \mathcal{A}'$ despite the fact that $\langle \mathcal{T}, \mathcal{A}' \rangle \models f$. This is impossible since \mathcal{A}' should be a maximal consistent subbase of \mathcal{A} . Since each $\circ_d \in \{\circ_{cl}, \circ_{card}, \circ_{rep}\}$ applied on closed a BBox preserves the closeness property, then clearly a composite modifier also preserves this closeness property.

The proof of item (2.b) follows immediately from the fact that i) $\forall \mathcal{A}_i \in \circ_{rep}(\mathcal{M})$, $\langle \mathcal{T}, \mathcal{A}_i \rangle$ is consistent, ii) if \mathcal{M} is consistent, then $\forall \circ_d \in \{\circ_{cl}, \circ_{card}, \circ_{rep}\}$ yields a consistent subbase, and iii) $\circ_{rep}(\mathcal{M}) = \mathcal{M}$ if \mathcal{M} is consistent. \square

Proof of Theorem 1. The proof relies on Lemma 1 (see also the explanations following Lemma 1 in the paper). \square

Justification of Figure 2 (Inclusion relations between composite modifiers): see following Proposition 6, Example 3, Proposition 7 and Example 4.

Proposition 6 (Part of the proof of Figure 2). Let $\mathcal{K}_{\mathcal{M}} = \langle \mathcal{T}, \mathcal{M} = \{\mathcal{A}\} \rangle$ be an inconsistent KB. Let $\{\mathcal{M}_1, \dots, \mathcal{M}_8\}$ be the MBoxes obtained by the eight composite modifiers $\{\circ_1, \dots, \circ_8\}$ summarized in Table 1. Then:

1. $\mathcal{M}_2 \subseteq \mathcal{M}_1$.
2. $\mathcal{M}_4 \subseteq \mathcal{M}_3$.
3. $\mathcal{M}_6 \subseteq \mathcal{M}_5$.
4. $\mathcal{M}_8 \subseteq \mathcal{M}_7$.
5. $\mathcal{M}_3 = \circ_{cl}(\mathcal{M}_2)$.
6. $\mathcal{M}_5 = \circ_{cl}(\mathcal{M}_1)$.
7. $\mathcal{M}_3 \subseteq \mathcal{M}_5$.
8. $\mathcal{M}_5 \subseteq_R \mathcal{M}_7$.

Proof.

- Items 1-4 follow from the definition of the elementary modifier \circ_{card} . Since \circ_{card} selects subsets of \mathcal{M} having maximal cardinality. Namely, given \mathcal{M} an MBox, we have $\circ_{card}(\mathcal{M}) \subseteq \mathcal{M}$. Hence relations $\mathcal{M}_4 \subseteq \mathcal{M}_3$, $\mathcal{M}_2 \subseteq \mathcal{M}_1$, $\mathcal{M}_6 \subseteq \mathcal{M}_5$, and $\mathcal{M}_8 \subseteq \mathcal{M}_7$ holds.
- Items 5-6 follow immediately from the definition of the elementary modifier \circ_{cl} , hence we trivially have $\mathcal{M}_5 = \circ_{cl}(\mathcal{M}_1)$ and $\mathcal{M}_3 = \circ_{cl}(\mathcal{M}_2)$.
- Let us show that $\mathcal{M}_2 \subseteq_{cl} \mathcal{M}_5$, namely $\forall A \in \mathcal{M}_2, \exists B \in \mathcal{M}_5$ such that $B = Cl(A)$. The proof is immediate. Recall that $\mathcal{M}_2 \subseteq \mathcal{M}_1$, hence $\forall A \in \mathcal{M}_2$ we also have $A \in \mathcal{M}_1$. Recall also that $\mathcal{M}_5 = \circ_{cl}(\mathcal{M}_1)$. This means that $\forall A \in \mathcal{M}_2, \exists B \in \mathcal{M}_5$ such that $B = Cl(A)$.
- Regarding the proof of Item 8, we have $\mathcal{M}_2 \subseteq_{cl} \mathcal{M}_5$. This means that $\forall A \in \mathcal{M}_2$, there exists $B \in \mathcal{M}_5$ such that $B = Cl(A)$. Said differently, $\forall A \in \mathcal{M}_2$, we have $Cl(A) \in \mathcal{M}_5$. Since $\mathcal{M}_3 = \circ_{cl}(\mathcal{M}_2)$, we conclude that $\mathcal{M}_3 \subseteq \mathcal{M}_5$.
- We now show that $\mathcal{M}_5 \subseteq_R \mathcal{M}_7$. Let $B \in \circ_{rep}(\{\mathcal{A}\})$ and let us show that there exists a set of assertions X such that $\circ_{cl}(\{B\}) \subseteq X$ and $X \in \mathcal{M}_7$. Since $B \in \circ_{rep}(\{\mathcal{A}\})$, this means by definition that $B \subseteq \mathcal{A}$ and hence $B \subseteq \circ_{cl}(\mathcal{A})$. Now, B is consistent, this means that there exists $R \in \circ_{rep}(\circ_{cl}(\mathcal{A})) = \mathcal{M}_7$ such that $B \subseteq R$. From Lemma 1, R is a closed set of assertions, then this means that $Cl(B) \subseteq R$.

Example 3 (Counter-examples showing that there are no reciprocal edges in Figure 2).

1. *The converse of $\mathcal{M}_2 \subseteq \mathcal{M}_1$ does not hold.*
 Let $\mathcal{T} = \{B \sqsubseteq C, C \sqsubseteq \neg D\}$ and $\mathcal{M} = \{\{B(a), C(a), D(a)\}\}$.
 It is easy to check that $\langle \mathcal{T}, \mathcal{M} \rangle$ is inconsistent. We have:
 $\mathcal{M}_1 = \circ_{rep}(\mathcal{M}) = \{\{C(a), B(a)\}, \{D(a)\}\}$, and
 $\mathcal{M}_2 = \circ_{card}(\mathcal{M}_1) = \{\{C(a), B(a)\}\}$.
 One can check that $\mathcal{M}_1 \not\subseteq_R \mathcal{M}_2$.
2. *The converse of $\mathcal{M}_4 \subseteq \mathcal{M}_3$ does not hold.*
 Let $\mathcal{T} = \{A \sqsubseteq B, B \sqsubseteq \neg C\}$ and $\mathcal{M} = \{\{A(a), C(a)\}\}$.
 It is easy to check that $\langle \mathcal{T}, \mathcal{M} \rangle$ is inconsistent. We have:
 $\mathcal{M}_1 = \circ_{rep}(\mathcal{M}) = \{\{A(a)\}, \{C(a)\}\}$,
 $\mathcal{M}_2 = \circ_{card}(\mathcal{M}_1) = \{\{A(a)\}, \{C(a)\}\}$,
 $\mathcal{M}_3 = \circ_{cl}(\mathcal{M}_2) = \{\{A(a), B(a)\}, \{C(a)\}\}$, and
 $\mathcal{M}_4 = \circ_{card}(\mathcal{M}_3) = \{\{A(a), B(a)\}\}$.
 One can check that $\mathcal{M}_3 \not\subseteq_R \mathcal{M}_4$.
3. *The converse of $\mathcal{M}_6 \subseteq \mathcal{M}_5$ does not hold.*
 Let $\mathcal{T} = \{B \sqsubseteq C, C \sqsubseteq \neg D\}$ and $\mathcal{M} = \{\{B(a), D(a)\}\}$.
 It is easy to check that $\langle \mathcal{T}, \mathcal{M} \rangle$ is inconsistent. We have:
 $\mathcal{M}_1 = \circ_{rep}(\mathcal{M}) = \{\{B(a)\}, \{D(a)\}\}$,
 $\mathcal{M}_5 = \circ_{cl}(\mathcal{M}_1) = \{\{C(a), B(a)\}, \{D(a)\}\}$, and
 $\mathcal{M}_6 = \circ_{card}(\mathcal{M}_5) = \{\{C(a), B(a)\}\}$.
 One can check that $\mathcal{M}_5 \not\subseteq_R \mathcal{M}_6$.
4. *The converse of $\mathcal{M}_8 \subseteq \mathcal{M}_7$ does not hold.*
 Let $\mathcal{T} = \{A \sqsubseteq B, B \sqsubseteq \neg D\}$ and $\mathcal{M} = \{\{A(a), D(a)\}\}$.
 It is easy to check that $\langle \mathcal{T}, \mathcal{M} \rangle$ is inconsistent. We have:
 $\circ_{cl}(\mathcal{M}) = \{\{A(a), B(a), D(a)\}\}$,
 $\mathcal{M}_7 = \circ_{rep}(\circ_{cl}(\mathcal{M})) = \{\{A(a), B(a)\}, \{D(a)\}\}$, and
 $\mathcal{M}_8 = \circ_{card}(\mathcal{M}_7) = \{\{A(a), B(a)\}\}$.
 One can check that $\mathcal{M}_7 \not\subseteq_R \mathcal{M}_8$.
5. *The converse of $\mathcal{M}_3 \subseteq \mathcal{M}_5$ does not hold.*
 Let $\mathcal{T} = \{A \sqsubseteq B, B \sqsubseteq C, C \sqsubseteq \neg D\}$ and $\mathcal{M} = \{\{A(a), B(a), D(a)\}\}$.
 It is easy to check that $\langle \mathcal{T}, \mathcal{M} \rangle$ is inconsistent. We have:
 $\mathcal{M}_1 = \circ_{rep}(\mathcal{M}) = \{\{A(a), B(a)\}, \{D(a)\}\}$,
 $\mathcal{M}_5 = \circ_{cl}(\mathcal{M}_1) = \{\{A(a), B(a), C(a)\}, \{D(a)\}\}$,
 $\mathcal{M}_2 = \circ_{card}(\mathcal{M}_1) = \{\{A(a), B(a)\}\}$, and
 $\mathcal{M}_3 = \circ_{cl}(\mathcal{M}_2) = \{\{A(a), B(a), C(a)\}\}$.
 One can check that $\mathcal{M}_5 \not\subseteq_R \mathcal{M}_3$.
6. *The converse of $\mathcal{M}_5 \subseteq_R \mathcal{M}_7$ does not hold.*
 Let $\mathcal{T} = \{A \sqsubseteq \neg B, B \sqsubseteq D\}$ and $\mathcal{M} = \{\{A(a), B(a)\}\}$.
 It is easy to check that $\langle \mathcal{T}, \mathcal{M} \rangle$ is inconsistent. We have:
 $\circ_{cl}(\mathcal{M}) = \{\{A(a), B(a), D(a)\}\}$,
 $\mathcal{M}_7 = \circ_{rep}(\circ_{cl}(\mathcal{M})) = \{\{A(a), D(a)\}, \{B(a), D(a)\}\}$,
 $\mathcal{M}_1 = \circ_{rep}(\mathcal{M}) = \{\{A(a)\}, \{B(a)\}\}$, and
 $\mathcal{M}_5 = \circ_{cl}(\mathcal{M}_1) = \{\{A(a)\}, \{B(a), D(a)\}\}$.
 One can check that $\mathcal{M}_7 \not\subseteq_R \mathcal{M}_5$.

Corollary 1. Let $\mathcal{K}_{\mathcal{M}} = \langle \mathcal{T}, \mathcal{M} = \{A\} \rangle$ be an inconsistent KB. Let $\{\mathcal{M}_1, \dots, \mathcal{M}_8\}$ be the MBoxes obtained by the eight composite modifiers $\{\circ_1, \dots, \circ_8\}$ summarized in Table 1. Then:

1. $\forall \mathcal{A}_i \in \mathcal{M}_3, \exists \mathcal{A}_j \in \mathcal{M}_1$ such that $\mathcal{A}_i = Cl(\mathcal{A}_j)$.
2. $\forall \mathcal{A}_i \in \mathcal{M}_4, \exists \mathcal{A}_j \in \mathcal{M}_1$ such that $\mathcal{A}_i = Cl(\mathcal{A}_j)$.
3. $\forall \mathcal{A}_i \in \mathcal{M}_6, \exists \mathcal{A}_j \in \mathcal{M}_1$ such that $\mathcal{A}_i = Cl(\mathcal{A}_j)$.
4. $\forall \mathcal{A}_i \in \mathcal{M}_1, \exists \mathcal{A}_j \in \mathcal{M}_7$ such that $\mathcal{A}_i \subseteq \mathcal{A}_j$.
5. $\forall \mathcal{A}_i \in \mathcal{M}_1, \exists \mathcal{A}_j \in \mathcal{M}_8$ such that $\mathcal{A}_i \subseteq \mathcal{A}_j$.
6. $\forall \mathcal{A}_i \in \mathcal{M}_4, \exists \mathcal{A}_j \in \mathcal{M}_2$ such that $\mathcal{A}_i = Cl(\mathcal{A}_j)$.
7. $\forall \mathcal{A}_i \in \mathcal{M}_2, \exists \mathcal{A}_j \in \mathcal{M}_7$ such that $\mathcal{A}_i \subseteq \mathcal{A}_j$.

8. $\forall \mathcal{A}_i \in \mathcal{M}_3, \exists \mathcal{A}_j \in \mathcal{M}_7$ such that $\mathcal{A}_i \subseteq \mathcal{A}_j$.
9. $\forall \mathcal{A}_i \in \mathcal{M}_4, \exists \mathcal{A}_j \in \mathcal{M}_7$ such that $\mathcal{A}_i \subseteq \mathcal{A}_j$.
10. $\forall \mathcal{A}_i \in \mathcal{M}_5, \exists \mathcal{A}_j \in \mathcal{M}_8$ such that $\mathcal{A}_i \subseteq \mathcal{A}_j$.

Proposition 7 (Part of the proof of Figure 2). *Let $\{\circ_1, \dots, \circ_8\}$ be the eight composite modifiers summarized in Table 1. Then:*

1. *There exists \mathcal{M} such that $\circ_6(\mathcal{M})$ and $\circ_8(\mathcal{M})$ are incomparable.*
2. *There exists \mathcal{M} such that $\circ_2(\mathcal{M})$ and $\circ_6(\mathcal{M})$ are incomparable.*
3. *There exists \mathcal{M} such that $\circ_3(\mathcal{M})$ and $\circ_6(\mathcal{M})$ are incomparable.*
4. *There exists \mathcal{M} such that $\circ_4(\mathcal{M})$ and $\circ_6(\mathcal{M})$ are incomparable.*
5. *There exists \mathcal{M} such that $\circ_2(\mathcal{M})$ and $\circ_8(\mathcal{M})$ are incomparable.*
6. *There exists \mathcal{M} such that $\circ_3(\mathcal{M})$ and $\circ_8(\mathcal{M})$ are incomparable.*
7. *There exists \mathcal{M} such that $\circ_4(\mathcal{M})$ and $\circ_8(\mathcal{M})$ are incomparable.*

Example 4 (Examples that prove Proposition 7). *The following examples prove the statements in Proposition 7.*

1. *There exists \mathcal{M} such that $\circ_6(\mathcal{M})$ and $\circ_8(\mathcal{M})$ are incomparable.*
Let $\mathcal{T} = \{B \sqsubseteq \neg C, B \sqsubseteq A, C \sqsubseteq A, A \sqsubseteq \neg D, D \sqsubseteq E, E \sqsubseteq F\}$ and $\mathcal{M} = \{\{A(a), B(a), C(a), D(a)\}\}$.
It is easy to check that $\langle \mathcal{T}, \mathcal{M} \rangle$ is inconsistent. We have:
 $\mathcal{M}_1 = \circ_{rep}(\mathcal{M}) = \{\{A(a), C(a)\}, \{A(a), B(a)\}, \{D(a)\}\}$, and
 $\mathcal{M}_5 = \circ_{cl}(\mathcal{M}_1) = \{\{A(a), C(a)\}, \{A(a), B(a)\}, \{D(a), E(a), F(a)\}\}$, and
 $\mathcal{M}_6 = \circ_{card}(\mathcal{M}_5) = \{\{D(a), E(a), F(a)\}\}$,
 $\circ_{cl}(\mathcal{M}) = \{\{A(a), B(a), C(a), D(a), E(a), F(a)\}\}$,
 $\mathcal{M}_7 = \circ_{rep}(\circ_{cl}(\mathcal{M})) = \{\{A(a), C(a), E(a), F(a)\}, \{A(a), B(a), E(a), F(a)\}, \{D(a), E(a), F(a)\}\}$, and
 $\mathcal{M}_8 = \circ_{rep}(\mathcal{M}_7) = \{\{A(a), C(a), E(a), F(a)\}, \{A(a), B(a), E(a), F(a)\}\}$
One can check that \mathcal{M}_6 and \mathcal{M}_8 are incomparable.
2. *There exists \mathcal{M} such that $\circ_2(\mathcal{M})$, $\circ_3(\mathcal{M})$ and $\circ_4(\mathcal{M})$ are incomparable with $\circ_6(\mathcal{M})$.*
Let $\mathcal{T} = \{A \sqsubseteq \neg B, C \sqsubseteq A, B \sqsubseteq D, D \sqsubseteq F\}$ and $\mathcal{M} = \{\{A(a), C(a), B(a)\}\}$.
It is easy to check that $\langle \mathcal{T}, \mathcal{M} \rangle$ is inconsistent. We have:
 $\mathcal{M}_1 = \circ_{rep}(\mathcal{M}) = \{\{A(a), C(a)\}, \{B(a)\}\}$,
 $\mathcal{M}_2 = \circ_{card}(\mathcal{M}_1) = \{\{A(a), C(a)\}\}$,
 $\mathcal{M}_5 = \circ_{cl}(\mathcal{M}_1) = \{\{A(a), C(a)\}, \{B(a), D(a), F(a)\}\}$,
 $\mathcal{M}_6 = \circ_{card}(\mathcal{M}_5) = \{\{B(a), D(a), F(a)\}\}$,
One can check that \mathcal{M}_2 is incomparable with \mathcal{M}_6 .
We have also $\mathcal{M}_2 = \mathcal{M}_3 = \mathcal{M}_4 = \{\{A(a), C(a)\}\}$. So, we conclude that \mathcal{M}_3 and \mathcal{M}_4 are incomparable with \mathcal{M}_6 .
3. *There exists \mathcal{M} such that $\circ_2(\mathcal{M})$, $\circ_3(\mathcal{M})$ and $\circ_4(\mathcal{M})$ are incomparable with $\circ_8(\mathcal{M})$.*
Let $\mathcal{T} = \{B \sqsubseteq A, C \sqsubseteq A, A \sqsubseteq \neg D, E \sqsubseteq D, D \sqsubseteq F\}$ and $\mathcal{M} = \{\{A(a), D(a), E(a)\}\}$.
It is easy to check that $\langle \mathcal{T}, \mathcal{M} \rangle$ is inconsistent. We have:
 $\mathcal{M}_1 = \circ_{rep}(\mathcal{M}) = \{\{A(a)\}, \{D(a), E(a)\}\}$, and
 $\mathcal{M}_2 = \circ_{card}(\mathcal{M}_1) = \{\{D(a), E(a)\}\}$, and
 $\mathcal{M}_3 = \mathcal{M}_4 = \{\{D(a), E(a), F(a)\}\}$,
 $\circ_{cl}(\mathcal{M}) = \{\{A(a), B(a), C(a), D(a), E(a), F(a)\}\}$,
 $\mathcal{M}_7 = \circ_{rep}(\circ_{cl}(\mathcal{M})) = \{\{A(a), C(a), B(a), F(a)\}, \{D(a), E(a), F(a)\}\}$, and
 $\mathcal{M}_8 = \circ_{rep}(\mathcal{M}_7) = \{\{A(a), C(a), B(a), F(a)\}\}$,
One can check that \mathcal{M}_2 , \mathcal{M}_3 and \mathcal{M}_4 are incomparable with \mathcal{M}_8 .

Proposition 8 (Proof of Equation 1 and Figure 3). *Let \mathcal{M} be a consistent MBox w.r.t. a TBox \mathcal{T} . Let q be a query. Then:*

1. *if $\langle \mathcal{T}, \mathcal{M} \rangle \models_{\cap} q$ then $\langle \mathcal{T}, \mathcal{M} \rangle \models_{\forall} q$.*
2. *if $\langle \mathcal{T}, \mathcal{M} \rangle \models_{\forall} q$ then $\langle \mathcal{T}, \mathcal{M} \rangle \models_{maj} q$.*
3. *if $\langle \mathcal{T}, \mathcal{M} \rangle \models_{maj} q$ then $\langle \mathcal{T}, \mathcal{M} \rangle \models_{\exists} q$.*

Proof of Proposition 8. Item 1 holds from the fact that $\forall \mathcal{A}_i \in \mathcal{M}$, we have $(\bigcap_{\mathcal{A}_i \in \mathcal{M}} \mathcal{A}_i) \subseteq \mathcal{A}_i$. Item 2 holds due to the fact that universal consequence requires that q follows from all ABoxes in \mathcal{M} . Hence, q holds in more than the half of \mathcal{A}_i 's in \mathcal{M} . Item 3 follows from the fact that a query is considered as valid using majority-based consequence relation if it is confirmed by more than the half of $\mathcal{A}_i \in \mathcal{M}$. Hence q follows from at least one ABox. \square

Finally, the following two lemmas about the cautiousness relation will be used later. Lemma 2 considers two MBoxes, with one included in the other. Lemma 3 considers two MBoxes, where one is the positive closure of the other.

Lemma 2. *Let \mathcal{M}_1 and \mathcal{M}_2 be two consistent MBoxes w.r.t. a TBox \mathcal{T} such that $\mathcal{M}_1 \subseteq \mathcal{M}_2$. Let q be a query. Then:*

1. *If $\langle \mathcal{T}, \mathcal{M}_2 \rangle \models_{\forall} q$ then $\langle \mathcal{T}, \mathcal{M}_1 \rangle \models_{\forall} q$.*
2. *If $\langle \mathcal{T}, \mathcal{M}_2 \rangle \models_{\cap} q$ then $\langle \mathcal{T}, \mathcal{M}_1 \rangle \models_{\cap} q$.*
3. *There are \mathcal{M}_1 and \mathcal{M}_2 such that the majority-based inference yields incomparable results.*

4. If $\langle \mathcal{T}, \mathcal{M}_1 \rangle \models_{\exists} q$ then $\langle \mathcal{T}, \mathcal{M}_2 \rangle \models_{\exists} q$.

Proof. The proof is immediate. For item 1, if q holds in all \mathcal{A}_i of \mathcal{M}_2 then trivially it holds in all \mathcal{A}_j of \mathcal{M}_1 (since $\mathcal{M}_1 \subseteq \mathcal{M}_2$). Item 2 holds due to the fact that $\mathcal{M}_1 \subseteq \mathcal{M}_2$ implies that $\bigcap_{\mathcal{A}_i \in \mathcal{M}_2} \mathcal{A}_i \subseteq \bigcap_{\mathcal{A}_j \in \mathcal{M}_1} \mathcal{A}_j$. Lastly, from item 4, if there exists an \mathcal{A}_i in \mathcal{M}_1 where q holds, then such \mathcal{A}_i also exists in \mathcal{M}_2 . \square

Example 5 (Counter-examples associated with Lemma 2). *The converse of Items 1 and 2 and 4 does not hold, as shown by the following counter-example. Let $\mathcal{T} = \emptyset$, $\mathcal{M}_1 = \{B(a)\}$ and $\mathcal{M}_2 = \{\{B(a)\}, \{B(c)\}, \{B(c)\}\}$. First, note that $\mathcal{M}_1 \subseteq \mathcal{M}_2$. Clearly $\langle \mathcal{T}, \mathcal{M}_1 \rangle \models_{\forall} B(a)$ (resp. $\langle \mathcal{T}, \mathcal{M}_1 \rangle \models_{\cap} B(a)$) holds, while $\langle \mathcal{T}, \mathcal{M}_2 \rangle \models_{\forall} B(a)$ (resp. $\langle \mathcal{T}, \mathcal{M}_2 \rangle \models_{\cap} B(a)$) does not hold. Similarly $\langle \mathcal{T}, \mathcal{M}_2 \rangle \models_{\exists} B(c)$ holds, while $\langle \mathcal{T}, \mathcal{M}_1 \rangle \models_{\exists} B(c)$ does not hold.*

Regarding majority-based inference, one can check that $\langle \mathcal{T}, \mathcal{M}_1 \rangle \models_{maj} B(a)$ holds while $\langle \mathcal{T}, \mathcal{M}_2 \rangle \models_{maj} B(a)$ does not hold. And $\langle \mathcal{T}, \mathcal{M}_2 \rangle \models_{maj} B(c)$ holds while $\langle \mathcal{T}, \mathcal{M}_1 \rangle \models_{maj} B(c)$ does not hold.

Lemma 3. *Let \mathcal{M}_1 and \mathcal{M}_2 be two consistent MBoxex w.r.t. \mathcal{T} . Let \mathcal{M}_2 be the positive closure of \mathcal{M}_1 . Let q be a Boolean query. Then:*

1. $\langle \mathcal{T}, \mathcal{M}_1 \rangle \models_{\forall} q$ iff $\langle \mathcal{T}, \mathcal{M}_2 \rangle \models_{\forall} q$.
2. $\langle \mathcal{T}, \mathcal{M}_1 \rangle \models_{maj} q$ iff $\langle \mathcal{T}, \mathcal{M}_2 \rangle \models_{maj} q$.
3. if $\langle \mathcal{T}, \mathcal{M}_1 \rangle \models_{\cap} q$ then $\langle \mathcal{T}, \mathcal{M}_2 \rangle \models_{\cap} q$.
4. $\langle \mathcal{T}, \mathcal{M}_1 \rangle \models_{\exists} q$ iff $\langle \mathcal{T}, \mathcal{M}_2 \rangle \models_{\exists} q$.

Proof of Lemma 3. The proof is again immediate. Items 1, 2 and 4 follow from the fact that, if \mathcal{A} is a consistent ABox with \mathcal{T} , then $\langle \mathcal{T}, \mathcal{A} \rangle \models q$ iff $\langle \mathcal{T}, Cl(\mathcal{A}) \rangle \models q$. Item 3 follows from the fact that $\mathcal{A}_i \subseteq Cl(\mathcal{A}_i)$ for each $\mathcal{A}_i \in \mathcal{M}_1$. Hence $\bigcap_{\mathcal{A}_i \in \mathcal{M}_1} \mathcal{A}_i \subseteq \bigcap_{\mathcal{A}_i \in \mathcal{M}_1} Cl(\mathcal{A}_i) = \bigcap_{\mathcal{A}_j \in \mathcal{M}_2} \mathcal{A}_j$. \square

Section 4: Comparison of Inconsistency-Tolerant Semantics w.r.t. Productivity

Proof of Figure 4 (intersection-based semantics) The relation pictured in the figure is proved by the following propositions and examples.

Proposition 9 (Proof of Figure 4, Part 1). *Let $\mathcal{K}_{\mathcal{M}} = \langle \mathcal{T}, \mathcal{M} = \{\mathcal{A}\} \rangle$ be an inconsistent KB. Let $\mathcal{M}_1, \dots, \mathcal{M}_8$ be the MBoxes obtained by applying the eight modifiers $\{\circ_1, \dots, \circ_8\}$, given in Table 1, on \mathcal{M} . Let q be a Boolean query. Then:*

1. If q is a safe conclusion of $\langle \mathcal{T}, \mathcal{M}_1 \rangle$ then q is a safe conclusion of $\langle \mathcal{T}, \mathcal{M}_2 \rangle$.
2. If q is a safe conclusion of $\langle \mathcal{T}, \mathcal{M}_1 \rangle$ then q is a safe conclusion of $\langle \mathcal{T}, \mathcal{M}_5 \rangle$.
3. If q is a safe conclusion of $\langle \mathcal{T}, \mathcal{M}_2 \rangle$ then q is a safe conclusion of $\langle \mathcal{T}, \mathcal{M}_3 \rangle$.
4. If q is a safe conclusion of $\langle \mathcal{T}, \mathcal{M}_3 \rangle$ then q is a safe conclusion of $\langle \mathcal{T}, \mathcal{M}_4 \rangle$.
5. If q is a safe conclusion of $\langle \mathcal{T}, \mathcal{M}_5 \rangle$ then q is a safe conclusion of $\langle \mathcal{T}, \mathcal{M}_3 \rangle$.
6. if q is a safe conclusion of $\langle \mathcal{T}, \mathcal{M}_5 \rangle$ then q is a safe conclusion of $\langle \mathcal{T}, \mathcal{M}_6 \rangle$.
7. If q is a safe conclusion of $\langle \mathcal{T}, \mathcal{M}_5 \rangle$ then q is a safe conclusion of $\langle \mathcal{T}, \mathcal{M}_7 \rangle$.
8. If q is a safe conclusion of $\langle \mathcal{T}, \mathcal{M}_7 \rangle$ then q is a safe conclusion of $\langle \mathcal{T}, \mathcal{M}_8 \rangle$.

Proof. The proof is as follows:

1. For items 1, we have $\mathcal{M}_2 \subseteq \mathcal{M}_1$, then following Item 2 of Lemma 2, if $\langle \mathcal{M}_1, \cap \rangle$ implies a query q then $\langle \mathcal{M}_2, \cap \rangle$ implies it also. The proof follow similarly for Items 4, 5, 6 and 8 since $\mathcal{M}_4 \subseteq \mathcal{M}_3$, $\mathcal{M}_3 \subseteq \mathcal{M}_5$, $\mathcal{M}_6 \subseteq \mathcal{M}_5$, and $\mathcal{M}_8 \subseteq \mathcal{M}_7$.
2. For items 2 and 3, we have $\mathcal{M}_5 = \circ_{cl}(\mathcal{M}_1)$ and $\mathcal{M}_3 = \circ_{cl}(\mathcal{M}_2)$. Then following Item 3 of Lemma 3, if a query holds in $\langle \mathcal{M}, \cap \rangle$ then it also holds in $\langle \circ_{cl}(\mathcal{M}), \cap \rangle$.
3. For item 7, we have $\forall A \in \mathcal{M}_5, \exists B \in \mathcal{M}_7$ such that $A \subseteq B$. Let $A(a) \in \bigcap_{\mathcal{A}_i \in \mathcal{M}_5} \mathcal{A}_i$. Then one can check that there is no conflict \mathcal{C} in $\langle \mathcal{T}, Cl(\mathcal{M}) \rangle$ such that $A(a) \in \mathcal{C}$. Indeed, assume that such conflict exists. Then this means that there exists $B(a) \in Cl(\mathcal{M})$ where $\langle \mathcal{T}, \{A(a), B(a)\} \rangle$ is conflicting. Two options:
 - i) $B(a) \in \mathcal{M}$. This means that there exists a maximally consistent subset X of \mathcal{M} with $B(a) \in X$. Since $B(a)$ is conflicting with $A(a)$, with respect to \mathcal{T} . Then $A(a)$ neither belongs to X nor to $Cl(X)$. This contradict the fact that $A(a) \in \bigcap_{\mathcal{A}_i \in \mathcal{M}_5} \mathcal{A}_i$.
 - ii) $B(a) \notin \mathcal{M}$. Let $Y \subseteq \mathcal{M}$ such that $\langle \mathcal{T}, Y \rangle \models B(a)$. Then clearly $\langle \mathcal{T}, Y \cup \{A(a)\} \rangle$ is inconsistent. Hence, there exists $D(a) \in \mathcal{M}$ such that $\langle \mathcal{T}, \{D(a), A(a)\} \rangle$ is conflicting and $D(a) \in Y$. This comes down to item (i). Now, since there is no conflict in $Cl(\mathcal{M})$ containing $A(a)$, then $A(a)$ belong to all maximally consistent subsets of $Cl(\mathcal{M})$, hence $A(a)$ belongs to $\bigcap_{\mathcal{A}_j \in \mathcal{M}_7} \mathcal{A}_j$. Therefore if q holds in $\langle \mathcal{M}_5, \cap \rangle$, then it holds that $\langle \mathcal{M}_5, \cap \rangle$.

\square

Example 6 (Proof of Figure 4, Part 2). *The following counter-examples show that no reciprocal edges hold in Figure 4.*

1. *There exists a KB, and a Boolean query q such that q is a safe conclusion of $\langle \mathcal{T}, \mathcal{M}_2 \rangle$, but q is not a safe conclusion of $\langle \mathcal{T}, \mathcal{M}_1 \rangle$:*
Let us consider $\mathcal{T} = \{A \sqsubseteq B, B \sqsubseteq \neg C\}$ and $\mathcal{M} = \{\{C(a), A(a), B(a)\}\}$.
It is easy to check that $\langle \mathcal{T}, \mathcal{M} \rangle$ is inconsistent. We have:
 $\mathcal{M}_1 = \{\{C(a)\}, \{A(a), B(a)\}\}$, and
 $\mathcal{M}_2 = \{\{A(a), B(a)\}\}$.
Let $q \leftarrow A(a)$ be a query. One can check that:
 $\mathcal{M}_2 \models_{\cap} q$, since $\bigcap_{\mathcal{A}_i \in \mathcal{M}_2} \mathcal{A}_i = \{A(a), B(a)\}$ but
 $\mathcal{M}_1 \not\models_{\cap} q$.
2. *There exists a KB, and a Boolean query q such that q is a safe conclusion of $\langle \mathcal{T}, \mathcal{M}_5 \rangle$, but q is not a safe conclusion of $\langle \mathcal{T}, \mathcal{M}_1 \rangle$:*
Let us consider $\mathcal{T} = \{B \sqsubseteq D, B \sqsubseteq \neg C, C \sqsubseteq D\}$ and $\mathcal{M} = \{\{C(a), B(a)\}\}$.
It is easy to check that $\langle \mathcal{T}, \mathcal{M} \rangle$ is inconsistent. We have:
 $\mathcal{M}_1 = \{\{C(a)\}, \{B(a)\}\}$, and
 $\mathcal{M}_5 = \{\{B(a), D(a)\}, \{C(a), D(a)\}\}$.
Let $q \leftarrow D(a)$ be a query. One can check that :
 $\mathcal{M}_5 \models_{\cap} q$ since $\bigcap_{\mathcal{A}_i \in \mathcal{M}_5} \mathcal{A}_i = \{D(a)\}$, but
 $\mathcal{M}_1 \not\models_{\cap} q$.
3. *There exists a KB, and a Boolean query q such that q is a safe conclusion of $\langle \mathcal{T}, \mathcal{M}_3 \rangle$, but q is not a safe conclusion of $\langle \mathcal{T}, \mathcal{M}_2 \rangle$:*
Let us consider $\mathcal{T} = \{B \sqsubseteq \neg C, C \sqsubseteq A, B \sqsubseteq A\}$ and $\mathcal{M} = \{\{C(a), B(a)\}\}$.
It is easy to check that $\langle \mathcal{T}, \mathcal{M} \rangle$ is inconsistent. We have:
 $\mathcal{M}_1 = \mathcal{M}_2 = \{\{C(a)\}, \{B(a)\}\}$, and
 $\mathcal{M}_3 = \{\{C(a), A(a)\}, \{B(a), A(a)\}\}$.
Let $q \leftarrow A(a)$ be a query. One can check that:
 $\mathcal{M}_3 \models_{\cap} q$, but
 $\mathcal{M}_2 \not\models_{\cap} q$.
4. *There exists a KB, and a Boolean query q such that q is a safe conclusion of $\langle \mathcal{T}, \mathcal{M}_4 \rangle$, but q is not a safe conclusion of $\langle \mathcal{T}, \mathcal{M}_3 \rangle$:*
Let us consider $\mathcal{T} = \{A \sqsubseteq B, B \sqsubseteq \neg D\}$ and $\mathcal{M} = \{\{A(a), D(a)\}\}$.
It is easy to check that $\langle \mathcal{T}, \mathcal{M} \rangle$ is inconsistent. We have:
 $\mathcal{M}_1 = \mathcal{M}_2 = \{\{A(a)\}, \{D(a)\}\}$,
 $\mathcal{M}_3 = \{\{A(a), B(a)\}, \{D(a)\}\}$, and
 $\mathcal{M}_4 = \{\{A(a), B(a)\}\}$.
Let $q \leftarrow A(a)$ be a query. One can check that
 $\mathcal{M}_4 \models_{\cap} q$ but
 $\mathcal{M}_3 \not\models_{\cap} q$.
5. *There exists a KB, and a Boolean query q such that q is a safe conclusion of $\langle \mathcal{T}, \mathcal{M}_3 \rangle$, but q is not a safe conclusion of $\langle \mathcal{T}, \mathcal{M}_5 \rangle$:*
Let us consider $\mathcal{T} = \{A \sqsubseteq B, B \sqsubseteq \neg D\}$ and $\mathcal{M} = \{\{A(a), D(a), B(a)\}\}$.
It is easy to check that $\langle \mathcal{T}, \mathcal{M} \rangle$ is inconsistent. We have:
 $\mathcal{M}_1 = \{\{A(a), B(a)\}, \{D(a)\}\}$,
 $\mathcal{M}_2 = \mathcal{M}_3 = \{\{A(a), B(a)\}\}$, and
 $\mathcal{M}_5 = \{\{A(a), B(a)\}, \{D(a)\}\}$.
Let $q \leftarrow A(a)$ be a query. One can check that
 $\mathcal{M}_3 \models_{\cap} q$ but
 $\mathcal{M}_5 \not\models_{\cap} q$.
6. *There exists a KB, and a Boolean query q such that q is a safe conclusion of $\langle \mathcal{T}, \mathcal{M}_6 \rangle$, but q is not a safe conclusion of \mathcal{M}_5 :*
Let us consider $\mathcal{T} = \{B \sqsubseteq C, C \sqsubseteq \neg D\}$ and $\mathcal{M} = \{\{B(a), D(a)\}\}$.
It is easy to check that $\langle \mathcal{T}, \mathcal{M} \rangle$ is inconsistent. We have:
 $\mathcal{M}_1 = \{\{B(a)\}, \{D(a)\}\}$,
 $\mathcal{M}_5 = \{\{B(a), C(a)\}, \{D(a)\}\}$, and
 $\mathcal{M}_6 = \{\{B(a), C(a)\}\}$.
Let $q \leftarrow B(a)$ be a query. One can check that
 $\mathcal{M}_6 \models_{\cap} q$ but
 $\mathcal{M}_5 \not\models_{\cap} q$.
7. *There exists a KB, and a Boolean query q such that q is a safe conclusion of $\langle \mathcal{T}, \mathcal{M}_7 \rangle$, but q is not a safe conclusion of \mathcal{M}_5 :*
Let $\mathcal{T} = \{A \sqsubseteq \neg B, B \sqsubseteq D\}$ and $\mathcal{M} = \{\{A(a), B(a)\}\}$.
It is easy to check that $\langle \mathcal{T}, \mathcal{M} \rangle$ is inconsistent. We have:
 $\mathcal{M}_1 = \{\{A(a)\}, \{B(a)\}\}$,
 $\mathcal{M}_5 = \{\{A(a)\}, \{B(a), D(a)\}\}$,
 $\circ_{cl}(\mathcal{M}) = \{\{A(a), B(a), D(a)\}\}$, and
 $\mathcal{M}_7 = \{\{A(a), D(a)\}, \{B(a), D(a)\}\}$.

Let $q \leftarrow D(a)$ be a query. One can deduce that:

$\mathcal{M}_7 \models_{\cap} q$ but

$\mathcal{M}_5 \not\models_{\cap} q$.

8. There exists a KB, and a Boolean query q such that q is a safe conclusion of $\langle \mathcal{T}, \mathcal{M}_8 \rangle$, but q is not a safe conclusion of $\langle \mathcal{T}, \mathcal{M}_7 \rangle$.
Let us consider $\mathcal{T} = \{A \sqsubseteq B, B \sqsubseteq \neg C, C \sqsubseteq D\}$ and $\mathcal{M} = \{\{A(a), C(a)\}\}$.

It is easy to check that $\langle \mathcal{T}, \mathcal{M} \rangle$ is inconsistent. We have:

$\circ_{cl}(\mathcal{M}) = \{A(a), C(a), B(a), D(a)\}$,

$\mathcal{M}_7 = \{\{A(a), B(a), D(a)\}, \{C(a), D(a)\}\}$, and

$\mathcal{M}_8 = \{\{A(a), B(a), D(a)\}\}$.

Let $q \leftarrow A(a)$ be a Boolean query. One can deduce that:

$\mathcal{M}_8 \models_{\cap} q$, but

$\mathcal{M}_7 \not\models_{\cap} q$.

Proposition 10 (Proof of Figure 4, Part 3). Let $\{\circ_1, \dots, \circ_8\}$ be the eight modifier given in Table 1. Let q be a Boolean query. Then:

1. There exists an MBox \mathcal{M} such that the safe inference from $\circ_2(\mathcal{M})$ is incomparable with the one obtained from $\circ_6(\mathcal{M})$.
2. There exists an MBox \mathcal{M} such that the safe inference from $\circ_3(\mathcal{M})$ is incomparable with the one obtained from $\circ_6(\mathcal{M})$.
3. There exists an MBox \mathcal{M} such that the safe inference from $\circ_6(\mathcal{M})$ is incomparable with the one obtained from $\circ_7(\mathcal{M})$.
4. There exists an MBox \mathcal{M} such that the safe inference from $\circ_6(\mathcal{M})$ is incomparable with the one obtained from $\circ_8(\mathcal{M})$.
5. There exists an MBox \mathcal{M} such that the safe inference from $\circ_2(\mathcal{M})$ is incomparable with the one obtained from $\circ_5(\mathcal{M})$.

Example 7. The following examples show the incomparabilities stated in the previous proposition.

1. The safe inference from \mathcal{M}_6 is incomparable with the one obtained from \mathcal{M}_7 .

Let $\mathcal{T} = \{C \sqsubseteq F, F \sqsubseteq A, A \sqsubseteq \neg B, B \sqsubseteq D\}$ and $\mathcal{M} = \{\{C(a), B(a)\}\}$.

It is easy to check that $\langle \mathcal{T}, \mathcal{M} \rangle$ is inconsistent. We have:

$\mathcal{M}_1 = \circ_{rep}(\mathcal{M}) = \{\{C(a)\}, \{B(a)\}\}$, and

$\mathcal{M}_5 = \circ_{cl}(\mathcal{M}_1) = \{\{A(a), C(a), F(a)\}, \{B(a), D(a)\}\}$, and

$\mathcal{M}_6 = \circ_{card}(\mathcal{M}_5) = \{\{A(a), C(a), F(a)\}\}$,

$\circ_{cl}(\mathcal{M}) = \{\{A(a), C(a), F(a), B(a), D(a)\}\}$,

$\mathcal{M}_7 = \circ_{rep}(\circ_{cl}(\mathcal{M})) = \{\{A(a), C(a), F(a), D(a)\}, \{D(a), B(a)\}\}$,

Let $q_1 \leftarrow F(a)$ and $q_2 \leftarrow D(a)$ be two queries. One can check that:

$\langle \mathcal{M}_7, \cap \rangle \models q_2$ but $\langle \mathcal{M}_6, \cap \rangle \not\models q_2$ while $\langle \mathcal{M}_6, \cap \rangle \models q_1$ but $\langle \mathcal{M}_7, \cap \rangle \not\models q_1$.

2. The safe inference from \mathcal{M}_6 is incomparable with the one obtained from \mathcal{M}_8 .

Let $\mathcal{T} = \{B \sqsubseteq \neg C, B \sqsubseteq A, C \sqsubseteq A, A \sqsubseteq \neg D, D \sqsubseteq E, E \sqsubseteq F\}$ and $\mathcal{M} = \{\{A(a), B(a), C(a), D(a)\}\}$.

It is easy to check that $\langle \mathcal{T}, \mathcal{M} \rangle$ is inconsistent. We have:

$\mathcal{M}_1 = \circ_{rep}(\mathcal{M}) = \{\{A(a), C(a)\}, \{A(a), B(a)\}, \{D(a)\}\}$,

$\mathcal{M}_5 = \circ_{cl}(\mathcal{M}_1) = \{\{A(a), C(a)\}, \{A(a), B(a)\}, \{D(a), E(a), F(a)\}\}$,

$\mathcal{M}_6 = \circ_{card}(\mathcal{M}_5) = \{\{D(a), E(a), F(a)\}\}$,

$\circ_{cl}(\mathcal{M}) = \{\{A(a), B(a), C(a), D(a), E(a), F(a)\}\}$,

$\mathcal{M}_7 = \circ_{rep}(\circ_{cl}(\mathcal{M})) = \{\{A(a), C(a), E(a), F(a)\}, \{A(a), B(a), E(a), F(a)\}, \{D(a), E(a), F(a)\}\}$, and

$\mathcal{M}_8 = \circ_{rep}(\mathcal{M}_7) = \{\{A(a), C(a), E(a), F(a)\}, \{A(a), B(a), E(a), F(a)\}\}$

Let $q_1 \leftarrow D(a)$ and $q_2 \leftarrow A(a)$ be two queries. One can check that:

$\langle \mathcal{M}_8, \cap \rangle \models q_2$ but $\langle \mathcal{M}_6, \cap \rangle \not\models q_2$ while $\langle \mathcal{M}_6, \cap \rangle \models q_1$ but $\langle \mathcal{M}_8, \cap \rangle \not\models q_1$.

3. The safe inference from \mathcal{M}_2 is incomparable with the one obtained from \mathcal{M}_6 .

Let $\mathcal{T} = \{A \sqsubseteq \neg B, C \sqsubseteq A, B \sqsubseteq D, D \sqsubseteq F\}$ and $\mathcal{M} = \{\{A(a), C(a), B(a)\}\}$.

It is easy to check that $\langle \mathcal{T}, \mathcal{M} \rangle$ is inconsistent. We have:

$\mathcal{M}_1 = \circ_{rep}(\mathcal{M}) = \{\{A(a), C(a)\}, \{B(a)\}\}$,

$\mathcal{M}_2 = \circ_{card}(\mathcal{M}_1) = \{\{A(a), C(a)\}\}$,

$\mathcal{M}_5 = \circ_{cl}(\mathcal{M}_1) = \{\{A(a), C(a)\}, \{B(a), D(a), F(a)\}\}$,

$\mathcal{M}_6 = \circ_{card}(\mathcal{M}_5) = \{\{B(a), D(a), F(a)\}\}$,

Let $q_1 \leftarrow A(a)$ and $q_2 \leftarrow B(a)$ be two queries. One can check that:

$\langle \mathcal{M}_2, \cap \rangle \models q_1$ but $\langle \mathcal{M}_6, \cap \rangle \not\models q_1$ while $\langle \mathcal{M}_6, \cap \rangle \models q_2$ but $\langle \mathcal{M}_2, \cap \rangle \not\models q_2$.

4. The safe inference from \mathcal{M}_2 is incomparable with the one obtained from \mathcal{M}_5 .

Let $\mathcal{T} = \{A \sqsubseteq B, C \sqsubseteq B, A \sqsubseteq \neg C, D \sqsubseteq C\}$ and $\mathcal{M} = \{\{A(a), C(a), D(a)\}\}$.

It is easy to check that $\langle \mathcal{T}, \mathcal{M} \rangle$ is inconsistent. We have:

$\mathcal{M}_1 = \circ_{rep}(\mathcal{M}) = \{\{A(a)\}, \{C(a), D(a)\}\}$,

$\mathcal{M}_2 = \circ_{card}(\mathcal{M}_1) = \{\{C(a), D(a)\}\}$, and

$\mathcal{M}_5 = \circ_{cl}(\mathcal{M}_1) = \{\{A(a), B(a)\}, \{B(a), D(a), C(a)\}\}$,

Let $q_1 \leftarrow D(a)$ and $q_2 \leftarrow B(a)$ be two queries. One can check that:

$\langle \mathcal{M}_2, \cap \rangle \models q_1$ but $\langle \mathcal{M}_5, \cap \rangle \not\models q_1$ while $\langle \mathcal{M}_5, \cap \rangle \models q_2$ but $\langle \mathcal{M}_2, \cap \rangle \not\models q_2$.

Proof of Figure 5 (universal semantics) The relation pictured in the figure is proved by the following propositions and examples.

Proposition 11 (Proof of Figure 5, Part 1). *Let $\mathcal{K}_{\mathcal{M}} = \langle \mathcal{T}, \mathcal{M} = \{A\} \rangle$ be an inconsistent KB. Let $\mathcal{M}_1, \dots, \mathcal{M}_8$ be the MBoxes obtained by applying the eight modifiers, given in Table 1, on \mathcal{M} . Let q be a Boolean query. Then:*

1. q is a universal conclusion of $\langle \mathcal{T}, \mathcal{M}_1 \rangle$ iff q is a universal conclusion of $\langle \mathcal{T}, \mathcal{M}_5 \rangle$.
2. q is a universal conclusion of $\langle \mathcal{T}, \mathcal{M}_2 \rangle$ iff q is a universal conclusion of $\langle \mathcal{T}, \mathcal{M}_3 \rangle$.

Proof. Item 1 and 2 follow from item 1 of Lemma 3 and the facts that $\mathcal{M}_5 = \circ_{cl}(\mathcal{M}_1)$ and $\mathcal{M}_3 = \circ_{cl}(\mathcal{M}_2)$. \square

Proposition 12 (Proof of Figure 5, Part 2). *Let $\mathcal{K}_{\mathcal{M}} = \langle \mathcal{T}, \mathcal{M} = \{A\} \rangle$ be an inconsistent KB. Let $\mathcal{M}_1, \dots, \mathcal{M}_8$ be the MBoxes obtained by applying the eight modifiers, given in Table 1, on \mathcal{M} . Let q be a Boolean query. Then:*

1. If q is a universal conclusion of $\langle \mathcal{T}, \mathcal{M}_1 \rangle$ (or $\langle \mathcal{T}, \mathcal{M}_5 \rangle$) then q is a universal conclusion of $\langle \mathcal{T}, \mathcal{M}_2 \rangle$.
2. If q is a universal conclusion of $\langle \mathcal{T}, \mathcal{M}_3 \rangle$ (or $\langle \mathcal{T}, \mathcal{M}_2 \rangle$) then q is a universal conclusion of $\langle \mathcal{T}, \mathcal{M}_4 \rangle$.
3. If q is a universal conclusion of $\langle \mathcal{T}, \mathcal{M}_1 \rangle$ (or $\langle \mathcal{T}, \mathcal{M}_5 \rangle$) then q is a universal conclusion of $\langle \mathcal{T}, \mathcal{M}_6 \rangle$.
4. If q is a universal conclusion of $\langle \mathcal{T}, \mathcal{M}_7 \rangle$ then q is a universal conclusion of $\langle \mathcal{T}, \mathcal{M}_8 \rangle$.
5. If q is a universal conclusion of $\langle \mathcal{T}, \mathcal{M}_1 \rangle$ (or $\langle \mathcal{T}, \mathcal{M}_5 \rangle$) then q is a universal conclusion of $\langle \mathcal{T}, \mathcal{M}_7 \rangle$.

Proof. For Items 1, 2, 3 and 4, we have $\mathcal{M}_2 \subseteq \mathcal{M}_1$, $\mathcal{M}_4 \subseteq \mathcal{M}_3$, $\mathcal{M}_6 \subseteq \mathcal{M}_5$ and $\mathcal{M}_8 \subseteq \mathcal{M}_7$. Then following Item 2 of Lemma 2, we have if $\langle \mathcal{T}, \mathcal{M}_1 \rangle \models_{\forall} q$ then $\langle \mathcal{T}, \mathcal{M}_2 \rangle \models_{\forall} q$. Similarly for $\mathcal{M}_4 \subseteq \mathcal{M}_3$, $\mathcal{M}_6 \subseteq \mathcal{M}_5$ and $\mathcal{M}_8 \subseteq \mathcal{M}_7$.

Finally, for item 5 recall first that $\langle \mathcal{M}_5, \forall \rangle \equiv \langle \mathcal{M}_1, \forall \rangle$ and $\forall A \in \mathcal{M}_5, \exists B \in \mathcal{M}_7$ such that $A \subseteq B$. Now let us show that $\forall B \in \mathcal{M}_7, \exists A \in \mathcal{M}_5$ such that $A \subseteq B$. Let $B \in \mathcal{M}_7 = \circ_{rep}(\circ_{cl}(\mathcal{M}))$. This means that $B \subseteq \circ_{cl}(\mathcal{M})$ and B is a maximally consistent subset. Let $C \in \circ_{rep}(\mathcal{M})$. This means that $C \subseteq \mathcal{M} \subseteq \circ_{cl}(\mathcal{M})$. Since C is also a maximally consistent subset then $C \subseteq B$. Now, recall that B is a closed set of assertion, then $A = Cl(C) \subseteq B$. Therefore we conclude that if a conclusion holds from \mathcal{M}_5 , then it holds from \mathcal{M}_7 . \square

Example 8 (Proof of Figure 5, Part 3). *The following counter-examples show that no reciprocal edges hold in Figure 5.*

1. *There exists a KB, and a Boolean query q such that q is a universal conclusion of $\langle \mathcal{T}, \mathcal{M}_2 \rangle$, but q is not a universal conclusion of $\langle \mathcal{T}, \mathcal{M}_1 \rangle$:*
Let us consider $\mathcal{T} = \{A \sqsubseteq B, B \sqsubseteq \neg C\}$ and $\mathcal{M} = \{\{A(a), B(a), C(a)\}\}$.
It is easy to check that $\langle \mathcal{T}, \mathcal{M} \rangle$ is inconsistent. We have:
 $\mathcal{M}_1 = \circ_{rep}(\mathcal{M}) = \{\{A(a), B(a)\}, \{C(a)\}\}$, and
 $\mathcal{M}_2 = \circ_{card}(\mathcal{M}_1) = \{\{A(a), B(a)\}\}$.
Let $q \leftarrow A(a)$ be a query. One can check that:
 $\langle \mathcal{M}_2, \forall \rangle \models q$ but
 $\langle \mathcal{M}_1, \forall \rangle \not\models q$, since $\langle \mathcal{T}, \{C(a)\} \rangle \not\models q$.
2. *There exists a KB, and a Boolean query q such that q is a universal conclusion of $\langle \mathcal{T}, \mathcal{M}_4 \rangle$, but q is not a universal conclusion of $\langle \mathcal{T}, \mathcal{M}_3 \rangle$:*
Let us consider $\mathcal{T} = \{A \sqsubseteq \neg B, A \sqsubseteq F\}$ and $\mathcal{M} = \{\{A(a), B(a)\}\}$.
It is easy to check that $\langle \mathcal{T}, \mathcal{M} \rangle$ is inconsistent. We have:
 $\mathcal{M}_1 = \mathcal{M}_2 = \{\{A(a)\}, \{B(a)\}\}$,
 $\mathcal{M}_3 = \{\{A(a), F(a)\}, \{B(a)\}\}$, and
 $\mathcal{M}_4 = \{\{A(a), F(a)\}\}$.
Let $q \leftarrow F(a)$ be a query. One can check that:
 $\langle \mathcal{M}_4, \forall \rangle \models q$ but
 $\langle \mathcal{M}_3, \forall \rangle \not\models q$, since $\langle \mathcal{T}, \{B(a)\} \rangle \not\models q$.
3. *There exists a KB, and a Boolean query q such that q is a universal conclusion of $\langle \mathcal{T}, \mathcal{M}_6 \rangle$, but q is not a universal conclusion of $\langle \mathcal{T}, \mathcal{M}_5 \rangle$:*
Let us consider $\mathcal{T} = \{B \sqsubseteq C, C \sqsubseteq \neg D\}$ and $\mathcal{M} = \{\{B(a), D(a)\}\}$.
It is easy to check that $\langle \mathcal{T}, \mathcal{M} \rangle$ is inconsistent. We have:
 $\mathcal{M}_1 = \{\{B(a)\}, \{D(a)\}\}$,
 $\mathcal{M}_5 = \{\{B(a), C(a)\}, \{D(a)\}\}$, and
 $\mathcal{M}_6 = \{\{B(a), C(a)\}\}$.
Let $q \leftarrow C(a)$ be a query. One can check that:
 $\langle \mathcal{M}_6, \forall \rangle \models q$ but
 $\langle \mathcal{M}_5, \forall \rangle \not\models q$, since $\langle \mathcal{T}, \{D(a)\} \rangle \not\models q$.

4. There exists a KB, and a Boolean query q such that q is a universal conclusion of $\langle \mathcal{T}, \mathcal{M}_8 \rangle$, but q is not a universal conclusion of $\langle \mathcal{T}, \mathcal{M}_7 \rangle$:
 Let us consider $\mathcal{T} = \{A \sqsubseteq B, B \sqsubseteq \neg C, C \sqsubseteq D, D \sqsubseteq F\}$ and $\mathcal{M} = \{\{A(a), C(a)\}\}$.
 It is easy to check that $\langle \mathcal{T}, \mathcal{M} \rangle$ is inconsistent. We have:
 $\circ_{cl}(\mathcal{M}) = \{A(a), C(a), B(a), D(a), F(a)\}$,
 $\mathcal{M}_7 = \{\{A(a), B(a), D(a), F(a)\}, \{C(a), D(a), F(a)\}\}$, and
 $\mathcal{M}_8 = \{\{A(a), B(a), D(a), F(a)\}\}$.
 Let $q \leftarrow A(a)$ be a query. One can check that:
 $\langle \mathcal{M}_8, \forall \rangle \models q$, but
 $\langle \mathcal{M}_7, \forall \rangle \not\models q$, since $\langle \mathcal{T}, \{C(a), D(a), F(a)\} \rangle \not\models q$.
5. There exists a KB, and a Boolean query q such that q is a universal conclusion of $\langle \mathcal{T}, \mathcal{M}_7 \rangle$, but q is not a universal conclusion of $\langle \mathcal{T}, \mathcal{M}_5 \rangle$:
 Let $\mathcal{T} = \{A \sqsubseteq \neg B, B \sqsubseteq D\}$ and $\mathcal{M} = \{\{A(a), B(a)\}\}$.
 It is easy to check that $\langle \mathcal{T}, \mathcal{M} \rangle$ is inconsistent. We have:
 $\mathcal{M}_1 = \{\{A(a)\}, \{B(a)\}\}$,
 $\circ_{cl}(\mathcal{M}) = \{\{A(a), B(a), D(a)\}\}$, and
 $\mathcal{M}_7 = \{\{A(a), D(a)\}, \{B(a), D(a)\}\}$.
 Let $q \leftarrow D(a)$ be a query. One can check that:
 $\langle \mathcal{M}_7, \forall \rangle \models q$ but
 $\langle \mathcal{M}_1, \forall \rangle \not\models q$, since $\langle \mathcal{T}, \{A(a)\} \rangle$.

Proposition 13 (Proof of Figure 5, Part 4). Let $\{\circ_1, \dots, \circ_8\}$ be the eight modifiers given in Table 1. Then:

1. There exists an MBox \mathcal{M} such that the universal inference from $\circ_6(\mathcal{M})$ is incomparable with the one obtained from $\circ_7(\mathcal{M})$.
2. There exists an MBox \mathcal{M} such that the universal inference from $\circ_6(\mathcal{M})$ is incomparable with the one obtained from $\circ_8(\mathcal{M})$.
3. There exists an MBox \mathcal{M} such that the universal inference from $\circ_2(\mathcal{M})$ (resp. $\circ_3(\mathcal{M})$, $\circ_4(\mathcal{M})$) is incomparable with the one obtained from $\circ_6(\mathcal{M})$.
4. There exists an MBox \mathcal{M} such that the universal inference from $\circ_2(\mathcal{M})$ (resp. $\circ_3(\mathcal{M})$, $\circ_4(\mathcal{M})$) is incomparable with the one obtained from $\circ_7(\mathcal{M})$.
5. There exists an MBox \mathcal{M} such that the universal inference from $\circ_2(\mathcal{M})$ (resp. $\circ_3(\mathcal{M})$, $\circ_4(\mathcal{M})$) is incomparable with the one obtained from $\circ_8(\mathcal{M})$.

Example 9. The following examples prove the incomparabilities stated in the previous proposition.

1. The universal inference from \mathcal{M}_6 is incomparable with the one obtained from \mathcal{M}_7 .
 Let $\mathcal{T} = \{C \sqsubseteq F, F \sqsubseteq A, A \sqsubseteq \neg B, B \sqsubseteq D\}$ and $\mathcal{M} = \{\{C(a), B(a)\}\}$.
 It is easy to check that $\langle \mathcal{T}, \mathcal{M} \rangle$ is inconsistent. We have:
 $\mathcal{M}_1 = \circ_{rep}(\mathcal{M}) = \{\{C(a)\}, \{B(a)\}\}$, and
 $\mathcal{M}_5 = \circ_{cl}(\mathcal{M}_1) = \{\{A(a), C(a), F(a)\}, \{B(a), D(a)\}\}$, and
 $\mathcal{M}_6 = \circ_{card}(\mathcal{M}_5) = \{\{A(a), C(a), F(a)\}\}$,
 $\circ_{cl}(\mathcal{M}) = \{\{A(a), C(a), F(a), B(a), D(a)\}\}$,
 $\mathcal{M}_7 = \circ_{rep}(\circ_{cl}(\mathcal{M})) = \{\{A(a), C(a), F(a), D(a)\}, \{D(a), B(a)\}\}$.
 Let $q_1 \leftarrow F(a)$ and $q_2 \leftarrow D(a)$ be two queries. One can check that:
 $\langle \mathcal{M}_7, \forall \rangle \models q_2$ but $\langle \mathcal{M}_6, \forall \rangle \not\models q_2$ while $\langle \mathcal{M}_6, \forall \rangle \models q_1$ but $\langle \mathcal{M}_7, \forall \rangle \not\models q_1$.
2. The universal inference from \mathcal{M}_6 is incomparable with the one obtained from \mathcal{M}_8 .
 Let $\mathcal{T} = \{B \sqsubseteq \neg C, B \sqsubseteq A, C \sqsubseteq A, A \sqsubseteq \neg D, D \sqsubseteq E, E \sqsubseteq F\}$ and $\mathcal{M} = \{\{A(a), B(a), C(a), D(a)\}\}$.
 It is easy to check that $\langle \mathcal{T}, \mathcal{M} \rangle$ is inconsistent. We have:
 $\mathcal{M}_1 = \circ_{rep}(\mathcal{M}) = \{\{A(a), C(a)\}, \{A(a), B(a)\}, \{D(a)\}\}$, and
 $\mathcal{M}_5 = \circ_{cl}(\mathcal{M}_1) = \{\{A(a), C(a)\}, \{A(a), B(a)\}, \{D(a), E(a), F(a)\}\}$, and
 $\mathcal{M}_6 = \circ_{card}(\mathcal{M}_5) = \{\{D(a), E(a), F(a)\}\}$,
 $\circ_{cl}(\mathcal{M}) = \{\{A(a), B(a), C(a), D(a), E(a), F(a)\}\}$,
 $\mathcal{M}_7 = \circ_{rep}(\circ_{cl}(\mathcal{M})) = \{\{A(a), C(a), E(a), F(a)\}, \{A(a), B(a), E(a), F(a)\}, \{D(a), E(a), F(a)\}\}$, and
 $\mathcal{M}_8 = \circ_{card}(\mathcal{M}_7) = \{\{A(a), C(a), E(a), F(a)\}, \{A(a), B(a), E(a), F(a)\}\}$.
 Let $q_1 \leftarrow D(a)$ and $q_2 \leftarrow A(a)$ be two queries. One can check that:
 $\langle \mathcal{M}_8, \forall \rangle \models q_2$ but $\langle \mathcal{M}_6, \forall \rangle \not\models q_2$ while $\langle \mathcal{M}_6, \forall \rangle \models q_1$ but $\langle \mathcal{M}_8, \forall \rangle \not\models q_1$.
3. The universal inference from \mathcal{M}_2 (resp. \mathcal{M}_3 and \mathcal{M}_4) is incomparable with the one obtained from \mathcal{M}_6 .
 Let $\mathcal{T} = \{A \sqsubseteq \neg B, C \sqsubseteq A, B \sqsubseteq D, D \sqsubseteq F\}$ and $\mathcal{M} = \{\{A(a), C(a), B(a)\}\}$.
 It is easy to check that $\langle \mathcal{T}, \mathcal{M} \rangle$ is inconsistent. We have:
 $\mathcal{M}_1 = \circ_{rep}(\mathcal{M}) = \{\{A(a), C(a)\}, \{B(a)\}\}$,
 $\mathcal{M}_2 = \circ_{card}(\mathcal{M}_2) = \{\{A(a), C(a)\}\}$,
 $\mathcal{M}_4 = \{\{A(a), C(a)\}\}$,
 $\mathcal{M}_5 = \circ_{cl}(\mathcal{M}_1) = \{\{A(a), C(a)\}, \{B(a), D(a), F(a)\}\}$,

$\mathcal{M}_6 = \circ_{card}(\mathcal{M}_5) = \{\{B(a), D(a), F(a)\}\},$

Let $q_1 \leftarrow A(a)$ and $q_2 \leftarrow B(a)$ be two queries. One can check that:

$\langle \mathcal{M}_2, \forall \rangle \models q_1$ but $\langle \mathcal{M}_6, \forall \rangle \not\models q_1$ while $\langle \mathcal{M}_6, \forall \rangle \models q_2$ but $\langle \mathcal{M}_2, \forall \rangle \not\models q_2$. Similarly for \mathcal{M}_4

4. The universal inference from \mathcal{M}_2 (resp. \mathcal{M}_3 and \mathcal{M}_4 is incomparable with the one obtained from \mathcal{M}_7 .

Let $\mathcal{T} = \{A \sqsubseteq \neg B, C \sqsubseteq A, B \sqsubseteq D, D \sqsubseteq F\}$ and $\mathcal{M} = \{\{A(a), C(a), B(a)\}\}$.

It is easy to check that $\langle \mathcal{T}, \mathcal{M} \rangle$ is inconsistent. We have:

$\mathcal{M}_1 = \circ_{rep}(\mathcal{M}) = \{\{A(a), C(a)\}, \{B(a)\}\},$

$\mathcal{M}_2 = \circ_{card}(\mathcal{M}_1) = \{\{A(a), C(a)\}\},$

$\mathcal{M}_4 = \{\{A(a), C(a)\}\},$

$\circ_{cl}(\mathcal{M}) = \{\{A(a), C(a), B(a), D(a), F(a)\}\},$

$\mathcal{M}_7 = \circ_{rep}(\circ_{cl}(\mathcal{M})) = \{\{A(a), C(a), D(a), F(a)\}, \{B(a), D(a), F(a)\}\},$

Let $q_1 \leftarrow A(a)$ and $q_2 \leftarrow D(a)$ be two queries. One can check that:

$\langle \mathcal{M}_2, \forall \rangle \models q_1$ but $\langle \mathcal{M}_7, \forall \rangle \not\models q_1$ while $\langle \mathcal{M}_7, \forall \rangle \models q_2$ but $\langle \mathcal{M}_2, \forall \rangle \not\models q_2$. Similarly for \mathcal{M}_4 .

5. The universal inference from \mathcal{M}_2 (resp. \mathcal{M}_3 and \mathcal{M}_4 is incomparable with the one obtained from \mathcal{M}_8 .

Let $\mathcal{T} = \{B \sqsubseteq A, C \sqsubseteq A, A \sqsubseteq \neg D, E \sqsubseteq D, D \sqsubseteq F\}$ and $\mathcal{M} = \{\{A(a), D(a), E(a)\}\}$.

It is easy to check that $\langle \mathcal{T}, \mathcal{M} \rangle$ is inconsistent. We have:

$\mathcal{M}_1 = \circ_{rep}(\mathcal{M}) = \{\{A(a)\}, \{D(a), E(a)\}\},$ and

$\mathcal{M}_2 = \circ_{card}(\mathcal{M}_1) = \{D(a), E(a)\},$ and

$\mathcal{M}_4 = \{D(a), E(a), F(a)\},$

$\circ_{cl}(\mathcal{M}) = \{\{A(a), B(a), C(a), D(a), E(a), F(a)\}\},$

$\mathcal{M}_7 = \circ_{rep}(\circ_{cl}(\mathcal{M})) = \{\{A(a), C(a), B(a), F(a)\}, \{D(a), E(a), F(a)\}\},$ and

$\mathcal{M}_8 = \circ_{rep}(\mathcal{M}_7) = \{\{A(a), C(a), B(a), F(a)\}\},$

Let $q_1 \leftarrow D(a)$ and $q_2 \leftarrow A(a)$ be two queries. One can check that:

$\langle \mathcal{M}_2, \forall \rangle \models q_1$ but $\langle \mathcal{M}_8, \forall \rangle \not\models q_1$ while $\langle \mathcal{M}_8, \forall \rangle \models q_2$ but $\langle \mathcal{M}_2, \forall \rangle \not\models q_2$. Similarly for \mathcal{M}_4

Proof of Figure 6 (majority-based semantics) The relation pictured in the figure is proved by the following propositions and examples.

Proposition 14 (Proof of Figure 6, Part 1). Let $\mathcal{K}_{\mathcal{M}} = \langle \mathcal{T}, \mathcal{M} = \{A\} \rangle$ be an inconsistent KB. Let $\mathcal{M}_1, \dots, \mathcal{M}_8$ be the MBoxes obtained by applying the eight modifiers, given in Table 1, on \mathcal{M} . Let q be a Boolean query. Then:

- $\langle \mathcal{T}, \mathcal{M}_1 \rangle \models_{maj} q$ iff $\langle \mathcal{T}, \mathcal{M}_5 \rangle \models_{maj} q$.
- $\langle \mathcal{T}, \mathcal{M}_2 \rangle \models_{maj} q$ iff $\langle \mathcal{T}, \mathcal{M}_3 \rangle \models_{maj} q$.
- If $\langle \mathcal{T}, \mathcal{M}_5 \rangle \models_{maj} q$ then $\langle \mathcal{T}, \mathcal{M}_7 \rangle \models_{maj} q$.

Proof. The proof of items 1 and 2 follow immediately from the proof of item 2 of Lemma 3, since $\mathcal{M}_5 = \circ_{cl}(\mathcal{M}_1)$ and $\mathcal{M}_2 = \circ_{cl}(\mathcal{M}_3)$. For Item 3, we have $\forall A_i \in \mathcal{M}_5, \exists A_j \in \mathcal{M}_7$ such that $A_i \subseteq A_j$. From proof of item 5 of proposition 12, we have $\forall A_j \in \mathcal{M}_7, \exists A_i \in \mathcal{M}_5$ such that $A_i \subseteq A_j$. We conclude that if a majority-based conclusion holds from \mathcal{M}_5 , it holds also from \mathcal{M}_7 . The converse does not hold. \square

Example 10 (Proof of Figure 6, Part 2). The following counter-examples show that no reciprocal edges hold in Figure 6.

1. There exists a KB, and a query q such that q is a majority-based conclusion of $\langle \mathcal{T}, \mathcal{M}_7 \rangle$, but q is not a majority-based conclusion of $\langle \mathcal{T}, \mathcal{M}_5 \rangle$:

Let $\mathcal{T} = \{A \sqsubseteq \neg B, B \sqsubseteq D\}$ and $\mathcal{M} = \{\{A(a), B(a)\}\}$.

It is easy to check that $\langle \mathcal{T}, \mathcal{M} \rangle$ is inconsistent. We have:

$\circ_{cl}(\mathcal{M}) = \{\{A(a), B(a), D(a)\}\},$

$\mathcal{M}_7 = \circ_{rep}(\circ_{cl}(\mathcal{M})) = \{\{A(a), D(a)\}, \{B(a), D(a)\}\},$

$\mathcal{M}_1 = \circ_{rep}(\mathcal{M}) = \{\{A(a)\}, \{B(a)\}\},$ and

$\mathcal{M}_5 = \circ_{cl}(\mathcal{M}_1) = \{\{A(a)\}, \{B(a), D(a)\}\},$

Let $q \leftarrow D(a)$ be a query. One can check that:

$\langle \mathcal{M}_7, maj \rangle \models q$ but $\langle \mathcal{M}_5, maj \rangle \not\models q$

Proposition 15 (Proof of Figure 6, Part 3). Let $\{\circ_1, \dots, \circ_8\}$ be the eight modifiers given in Table 1. Let q be a Boolean query. Then:

- There exists an MBox \mathcal{M} consistent w.r.t. \mathcal{T} such that the majority-based inference from $\langle \mathcal{T}, \circ_1(\mathcal{M}) \rangle$ is incomparable with the one obtained from $\langle \mathcal{T}, \circ_2(\mathcal{M}) \rangle$.
- There exists an MBox \mathcal{M} consistent w.r.t. \mathcal{T} such that the majority-based inference from $\langle \mathcal{T}, \circ_3(\mathcal{M}) \rangle$ is incomparable with the one obtained from $\langle \mathcal{T}, \circ_4(\mathcal{M}) \rangle$.
- There exists an MBox \mathcal{M} consistent w.r.t. \mathcal{T} such that the majority-based inference from $\langle \mathcal{T}, \circ_5(\mathcal{M}) \rangle$ is incomparable with the one obtained from $\langle \mathcal{T}, \circ_6(\mathcal{M}) \rangle$.
- There exists an MBox \mathcal{M} consistent w.r.t. \mathcal{T} such that the majority-based inference from $\langle \mathcal{T}, \circ_7(\mathcal{M}) \rangle$ is incomparable with the one obtained from $\langle \mathcal{T}, \circ_8(\mathcal{M}) \rangle$.

Example 11. The following examples show the incomparabilities stated in the previous proposition.

1. The majority-based inference from $\langle \mathcal{T}, \mathcal{M}_1 \rangle$ is incomparable with the one obtained from $\langle \mathcal{T}, \mathcal{M}_2 \rangle$.
 Let $\mathcal{T} = \{B \sqsubseteq \neg C, B \sqsubseteq A, C \sqsubseteq A, A \sqsubseteq \neg D, D \sqsubseteq E, E \sqsubseteq F\}$ and $\mathcal{M} = \{\{A(a), B(a), C(a), D(a), E(a), F(a)\}\}$.
 It is easy to check that $\langle \mathcal{T}, \mathcal{M} \rangle$ is inconsistent. We have:
 $\mathcal{M}_1 = \circ_{rep}(\mathcal{M}) = \{\{A(a), C(a)\}, \{A(a), B(a)\}, \{D(a), E(a), F(a)\}\}$, and
 $\mathcal{M}_2 = \circ_{card}(\mathcal{M}_1) = \{\{D(a), E(a), F(a)\}\}$
 Let $q_1 \leftarrow D(a)$ and $q_2 \leftarrow A(a)$ be two queries. One can check that:
 $\langle \mathcal{M}_1, maj \rangle \models q_2$ but $\langle \mathcal{M}_2, maj \rangle \not\models q_2$ while $\langle \mathcal{M}_2, maj \rangle \models q_1$ but $\langle \mathcal{M}_1, maj \rangle \not\models q_1$.
2. The majority-based inference from $\langle \mathcal{T}, \mathcal{M}_3 \rangle$ is incomparable with the one obtained from $\langle \mathcal{T}, \mathcal{M}_4 \rangle$.
 Let $\mathcal{T} = \{B \sqsubseteq \neg C, B \sqsubseteq A, C \sqsubseteq A, A \sqsubseteq \neg D, F \sqsubseteq D, D \sqsubseteq E\}$ and $\mathcal{M} = \{\{A(a), B(a), C(a), F(a), D(a)\}\}$.
 It is easy to check that $\langle \mathcal{T}, \mathcal{M} \rangle$ is inconsistent. We have:
 $\mathcal{M}_1 = \mathcal{M}_2 = \{\{A(a), C(a)\}, \{A(a), B(a)\}, \{D(a), F(a)\}\}$,
 $\mathcal{M}_3 = \{\{A(a), C(a)\}, \{A(a), B(a)\}, \{D(a), F(a), E(a)\}\}$,
 $\mathcal{M}_4 = \{\{D(a), E(a), F(a)\}\}$
 Let $q_1 \leftarrow D(a)$ and $q_2 \leftarrow A(a)$ be two queries. One can check that:
 $\langle \mathcal{M}_3, maj \rangle \models q_2$ but $\langle \mathcal{M}_4, maj \rangle \not\models q_2$ while $\langle \mathcal{M}_4, maj \rangle \models q_1$ but $\langle \mathcal{M}_3, maj \rangle \not\models q_1$.
3. The majority-based inference from $\langle \mathcal{T}, \mathcal{M}_5 \rangle$ is incomparable with the one obtained from $\langle \mathcal{T}, \mathcal{M}_6 \rangle$.
 Let $\mathcal{T} = \{B \sqsubseteq \neg C, B \sqsubseteq A, C \sqsubseteq A, A \sqsubseteq \neg D, F \sqsubseteq D, D \sqsubseteq E\}$ and $\mathcal{M} = \{\{A(a), B(a), C(a), F(a), D(a), E(a)\}\}$.
 It is easy to check that $\langle \mathcal{T}, \mathcal{M} \rangle$ is inconsistent. We have:
 $\mathcal{M}_1 = \mathcal{M}_5 = \{\{A(a), C(a)\}, \{A(a), B(a)\}, \{D(a), F(a), E(a)\}\}$,
 $\mathcal{M}_6 = \{\{D(a), E(a), F(a)\}\}$
 Let $q_1 \leftarrow D(a)$ and $q_2 \leftarrow A(a)$ be two queries. One can check that:
 $\langle \mathcal{M}_5, maj \rangle \models q_2$ but $\langle \mathcal{M}_6, maj \rangle \not\models q_2$ while $\langle \mathcal{M}_6, maj \rangle \models q_1$ but $\langle \mathcal{M}_5, maj \rangle \not\models q_1$.
4. The majority-based inference from $\langle \mathcal{T}, \mathcal{M}_7 \rangle$ is incomparable with the one obtained from $\langle \mathcal{T}, \mathcal{M}_8 \rangle$.
 Let $\mathcal{T} = \{B \sqsubseteq \neg C, B \sqsubseteq A, C \sqsubseteq A, A \sqsubseteq \neg D, F \sqsubseteq D, E \sqsubseteq D\}$ and $\mathcal{M} = \{\{A(a), F(a), E(a), B(a), C(a)\}\}$.
 It is easy to check that $\langle \mathcal{T}, \mathcal{M} \rangle$ is inconsistent. We have:
 $\circ_{cl}(\mathcal{M}) = \{\{A(a), C(a), B(a), D(a), F(a), E(a)\}\}$,
 $\mathcal{M}_7 = \{\{D(a), E(a), F(a)\}, \{A(a), B(a)\}, \{A(a), C(a)\}\}$, and
 $\mathcal{M}_8 = \{\{D(a), E(a), F(a)\}\}$, and
 Let $q_1 \leftarrow D(a)$ and $q_2 \leftarrow A(a)$ be two queries. One can check that:
 $\langle \mathcal{M}_7, maj \rangle \models q_2$ but $\langle \mathcal{M}_8, maj \rangle \not\models q_2$ while $\langle \mathcal{M}_8, maj \rangle \models q_1$ but $\langle \mathcal{M}_7, maj \rangle \not\models q_1$.

Proof of Figure 7 (existential semantics) The relation pictured in the figure is proved by the following propositions and examples.

Proposition 16 (Proof of Figure 7, Part 1). Let $\mathcal{K}_{\mathcal{M}} = \langle \mathcal{T}, \mathcal{M} = \{A\} \rangle$ be an inconsistent DL-Lite KB. Let $\mathcal{M}_1, \dots, \mathcal{M}_8$ be the eight MBoxes given in Figure 1 and in Table 1. Let q be a Boolean query. Then:

1. if q is an existential conclusion of $\langle \mathcal{T}, \mathcal{M}_4 \rangle$ then q is an existential conclusion of $\langle \mathcal{T}, \mathcal{M}_3 \rangle$.
2. q is an existential conclusion of $\langle \mathcal{T}, \mathcal{M}_3 \rangle$ iff q is an existential conclusion of $\langle \mathcal{T}, \mathcal{M}_2 \rangle$.
3. if q is an existential conclusion of $\langle \mathcal{T}, \mathcal{M}_2 \rangle$ then q is an existential conclusion of $\langle \mathcal{T}, \mathcal{M}_1 \rangle$.
4. q is an existential conclusion of $\langle \mathcal{T}, \mathcal{M}_1 \rangle$ iff q is an existential conclusion of $\langle \mathcal{T}, \mathcal{M}_5 \rangle$.
5. if q is an existential conclusion of $\langle \mathcal{T}, \mathcal{M}_2 \rangle$ then q is an existential conclusion of $\langle \mathcal{T}, \mathcal{M}_5 \rangle$.
6. if q is an existential conclusion of $\langle \mathcal{T}, \mathcal{M}_6 \rangle$ the q is an existential conclusion of $\langle \mathcal{T}, \mathcal{M}_5 \rangle$.
7. if q is an existential conclusion of $\langle \mathcal{T}, \mathcal{M}_8 \rangle$ the q is an existential conclusion of $\langle \mathcal{T}, \mathcal{M}_7 \rangle$.
8. if q is an existential conclusion of $\langle \mathcal{T}, \mathcal{M}_5 \rangle$ the q is an existential conclusion of $\langle \mathcal{T}, \mathcal{M}_7 \rangle$.

Proof. Items 1, 3, 6, and 7 follow from item 4 of Lemma 2. Items 2 and 4 follow from Item 4 of Lemma 3. Items 5 and 8 hold due the fact that $\circ_2 \subseteq_{cl} \circ_5$ and $\circ_5 \subseteq_{cl} \circ_7$. \square

Example 12 (Proof of Figure 7, Part 2). The following examples show that the reciprocal edges do not hold in Figure 7. We do not include the examples that prove that all incomparabilities hold, since they are similar.

1. There exists an existential conclusion of $\langle \mathcal{T}, \mathcal{M}_4 \rangle$ which is not an existential conclusion of $\langle \mathcal{T}, \mathcal{M}_3 \rangle$:
 Let us consider $\mathcal{T} = \{A \sqsubseteq B, B \sqsubseteq C, C \sqsubseteq \neg D, D \sqsubseteq F\}$ and $\mathcal{M} = \{A(a), D(a)\}$. We have:
 $\mathcal{M}_1 = \mathcal{M}_2 = \{\{A(a)\}, \{D(a)\}\}$,
 $\mathcal{M}_3 = \{\{A(a), B(a), C(a)\}, \{D(a), F(a)\}\}$, and
 $\mathcal{M}_4 = \{\{A(a), B(a), D(a)\}\}$.
 Let $q \leftarrow D(a) \wedge F(a)$ be a Boolean query. We have:
 $\mathcal{M}_3 \models q$, since $\langle \mathcal{T}, \{D(a), F(a)\} \rangle \models q$. However $\mathcal{M}_4 \not\models q$.

2. There exists an existential conclusion of $\langle \mathcal{T}, \mathcal{M}_1 \rangle$ which is not an existential conclusion of $\langle \mathcal{T}, \mathcal{M}_2 \rangle$:
Let us consider $\mathcal{T} = \{A \sqsubseteq B, B \sqsubseteq \neg C, C \sqsubseteq D\}$ and $\mathcal{M} = \{A(a), B(a), C(a), D(a)\}$. We have:
 $\mathcal{M}_1 = \{\{A(a), B(a), D(a)\}, \{C(a), D(a)\}\}$, and
 $\mathcal{M}_2 = \{\{A(a), B(a), D(a)\}\}$.
Let $q \leftarrow C(a) \wedge D(a)$ be a Boolean query. One can easily check that $j\mathcal{M}_1, \exists_i \models q$ but $j\mathcal{M}_2, \exists_i \not\models q$.
3. There exists an existential conclusion of $\langle \mathcal{T}, \mathcal{M}_5 \rangle$ which is not an existential conclusion of $\langle \mathcal{T}, \mathcal{M}_2 \rangle$:
Let us consider $\mathcal{T} = \{A \sqsubseteq B, B \sqsubseteq C, C \sqsubseteq \neg D, D \sqsubseteq F\}$ and $\mathcal{M} = \{A(a), B(a), D(a)\}$. We have:
 $\mathcal{M}_1 = \{\{A(a), B(a)\}, \{D(a)\}\}$,
 $\mathcal{M}_5 = \{\{A(a), B(a), C(a)\}, \{D(a), F(a)\}\}$, and
 $\mathcal{M}_2 = \{\{A(a), B(a)\}\}$.
Let $q \leftarrow D(a) \wedge F(a)$ be a Boolean query. One can deduce that:
 $j\mathcal{M}_5, \exists_i \models q$, but
 $j\mathcal{M}_2, \exists_i \not\models q$.
4. There exists an existential conclusion of $\langle \mathcal{T}, \mathcal{M}_5 \rangle$ which is not an existential conclusion of $\langle \mathcal{T}, \mathcal{M}_6 \rangle$:
Let us consider $\mathcal{T} = \{A \sqsubseteq B, B \sqsubseteq C, C \sqsubseteq \neg D, D \sqsubseteq F\}$ and $\mathcal{M} = \{A(a), D(a)\}$. We have:
 $\mathcal{M}_1 = \{\{A(a)\}, \{D(a)\}\}$,
 $\mathcal{M}_5 = \{\{A(a), B(a), C(a)\}, \{D(a), F(a)\}\}$, and
 $\mathcal{M}_6 = \{\{A(a), B(a), C(a)\}\}$.
Let $q \leftarrow D(a) \wedge F(a)$ be a Boolean query. One can check that:
 $j\mathcal{M}_5, \exists_i \models q$, but
 $j\mathcal{M}_6, \exists_i \not\models q$.
5. There exists an existential conclusion of $\langle \mathcal{T}, \mathcal{M}_7 \rangle$ which is not an existential conclusion of $\langle \mathcal{T}, \mathcal{M}_8 \rangle$:
Let us consider $\mathcal{T} = \{A \sqsubseteq B, B \sqsubseteq \neg C, C \sqsubseteq D, D \sqsubseteq F\}$ and $\mathcal{M} = \{A(a), C(a)\}$. We have:
 $\circ_{cl}(\mathcal{M}) = \{A(a), C(a), B(a), D(a), F(a)\}$,
 $\mathcal{M}_7 = \{\{A(a), B(a), D(a), F(a)\}, \{C(a), D(a), F(a)\}\}$, and
 $\mathcal{M}_8 = \{\{A(a), B(a), D(a), F(a)\}\}$.
Let $q \leftarrow C(a) \wedge D(a)$ be a Boolean query. One can check that:
 $j\mathcal{M}_7, \exists_i \models q$, but
 $j\mathcal{M}_8, \exists_i \not\models q$.

Theorem 2 [Productivity of semantics] The inclusion relation \sqsubseteq is the smallest relation that contains the inclusions $\langle \circ_i, s_k \rangle \sqsubseteq \langle \circ_j, s_k \rangle$ defined by Propositions 1-4 and satisfying the two following conditions:

1. for all s_j, s_p and \circ_i , if $s_j \leq s_p$ then $\langle \circ_i, s_j \rangle \sqsubseteq \langle \circ_i, s_p \rangle$.
2. it is transitive.

Proof. The first point follows from the definition of \leq . Let $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ and $\mathcal{K} \models_{\langle \circ_i, s_j \rangle} q$. This means that $\langle \mathcal{T}, \circ_i(\mathcal{A}) \rangle \models_{s_j} q$. Since $s_j \leq s_p$, we have $\langle \mathcal{T}, \circ_i(\mathcal{A}) \rangle \models_{s_p} q$, hence $\mathcal{K} \models_{\langle \circ_i, s_p \rangle} q$. The transitivity of \sqsubseteq follows from its definition. Indeed, consider three semantics S_1, S_2, S_3 such that $S_1 \sqsubseteq S_2 \sqsubseteq S_3$, then $\forall k, \forall q$, if $\mathcal{K} \models_{S_1} q$ then $\mathcal{K} \models_{S_2} q$ and $\mathcal{K} \models_{S_3} q$. Hence $S_1 \sqsubseteq S_3$. The following Lemmas 4 and 5 show that there are no other inclusions: the first lemma states that a semantics cannot be included into another semantics with a strictly more cautious inference strategy; the second lemma states that any inclusion from a semantics to another with a strictly less cautious inference strategy can only be obtained by transitivity using an edge “internal” to the latter inference strategy. \square

Lemma 4. For all $\langle \circ_i, s_j \rangle$ and $\langle \circ_k, s_p \rangle$, if $s_p < s_j$ then $\langle \circ_i, s_j \rangle \not\sqsubseteq \langle \circ_k, s_p \rangle$:

Proof. To prove this lemma, we consider the following example. Let $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ with
 $\mathcal{A} = \{p_a(f, a), p_b(f, b), p_c(f, c), p_d(f, d), p_e(f, e)\}$ and
 $\mathcal{T} = \{p_a(Z, X), p_b(Z, Y) \rightarrow \perp; p_b(Z, X), p_c(Z, Y) \rightarrow \perp; p_b(Z, X), p_d(Z, Y) \rightarrow \perp; p_c(Z, X), p_d(Z, Y) \rightarrow \perp, p_a(Z, X), p_e(Z, Y) \rightarrow \perp\}$. We have $\circ_1(\mathcal{A})$ contains $\{p_a(f, a), p_c(f, c)\}$, $\{p_a(f, a), p_d(f, d)\}$, $\{p_b(f, b), p_e(f, e)\}$, $\{p_e(f, e), p_d(f, d)\}$ and $\{p_e(f, e), p_c(f, c)\}$. Since \mathcal{T} contains only negative constraints and all ABoxes in $\circ_1(\mathcal{A})$ have the same size, then we have $\circ_1(\mathcal{A}) = \circ_2(\mathcal{A}) = \circ_3(\mathcal{A}) = \dots = \circ_8(\mathcal{A})$.
One can check that $\circ_1(\mathcal{A}) \models_{\exists} p_a(f, a)$, but $\circ_1(\mathcal{A}) \not\models_X p_a(f, a)$, for $X \in \{maj, \forall, \cap\}$, thus there is no $\langle \circ_i, \exists \rangle \sqsubseteq \langle \circ_k, s_p \rangle$ for $s_p \in \{maj, \forall, \cap\}$. Similarly, we have $\circ_A \models_{maj} p_e(f, e)$, but $\circ_1(\mathcal{A}) \not\models_X p_e(f, e)$, for $X \in \{\forall, \cap\}$, thus there is no $\langle \circ_i, maj \rangle \sqsubseteq \langle \circ_k, s_p \rangle$ for $s_p \in \{\forall, \cap\}$.

Finally, by adding to the previous examples the five following rules: $p_a(X, Y) \rightarrow p(X, Z), \dots, p_e(X, Y) \rightarrow p(X, Z)$ (which produce non-ground atoms), we do not change the repairs, hence we still have the property $\circ_1(\mathcal{A}) = \circ_2(\mathcal{A}) = \circ_3(\mathcal{A}) = \dots = \circ_8(\mathcal{A})$. Furthermore, we have $\circ_1(\mathcal{A}) \models_{\forall} \exists X \exists Y p(X, Y)$, but $\circ_1(\mathcal{A}) \not\models_{\cap} \exists X \exists Y p(X, Y)$, thus there is no $\langle \circ_i, \forall \rangle \sqsubseteq \langle \circ_k, \cap \rangle$. \square

Lemma 5. For all $\langle \circ_i, s_j \rangle$ and $\langle \circ_k, s_p \rangle$, if $\langle \circ_i, s_j \rangle \sqsubseteq \langle \circ_k, s_p \rangle$ and $s_j < s_p$, then $\langle \circ_i, s_p \rangle \sqsubseteq \langle \circ_k, s_p \rangle$.

To prove this lemma, we did not find a “generic” example as in the previous proof, hence we checked all cases one by one. Examples showing the incomparability can easily be found (similarly to what has been done for the proofs in the preceding section).

Note that when we restrict queries to ground atoms additional inclusions hold. We did not consider this specific, nevertheless important, case in the paper for space restriction reasons.

Finally, the following schema pictures all inclusions between semantics.